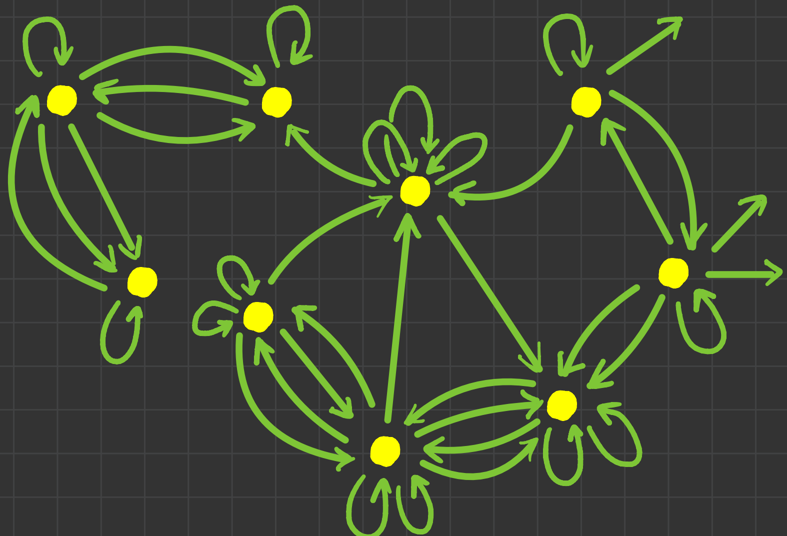
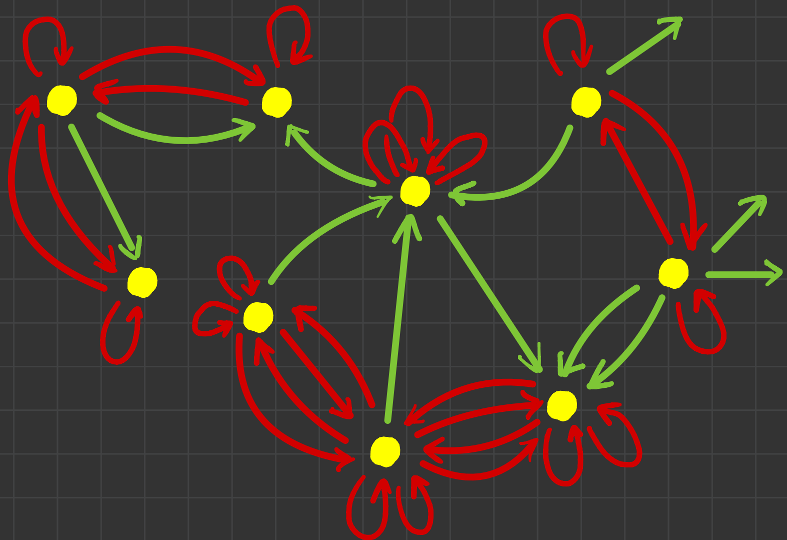
Moduli Set Perspective

We start with some objects and maps between them…



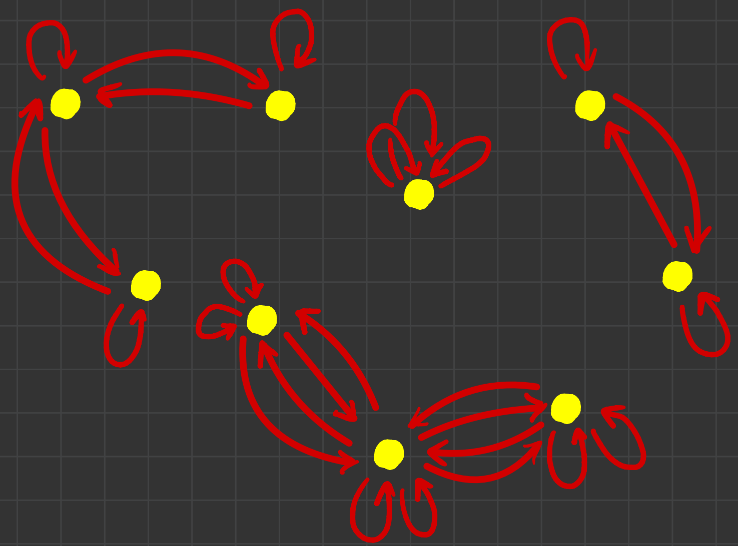
Moduli Set Perspective

We then define a notion of “sameness” between these objects…



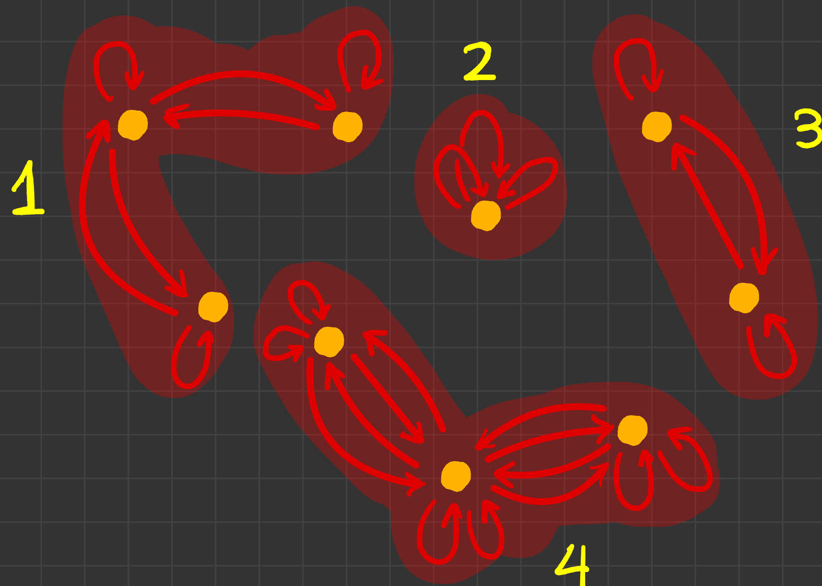
Moduli Set Perspective

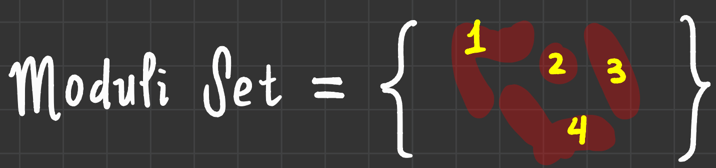
We "forget" all about maps…



Moduli Set Perspective

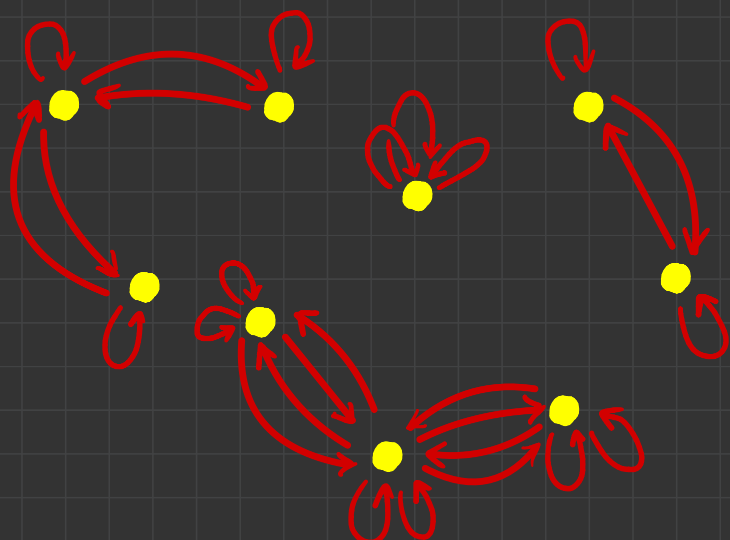
and we also “forget” all the different ways in which things are equivalent…





Moduli G r o u p o i d s 🥳

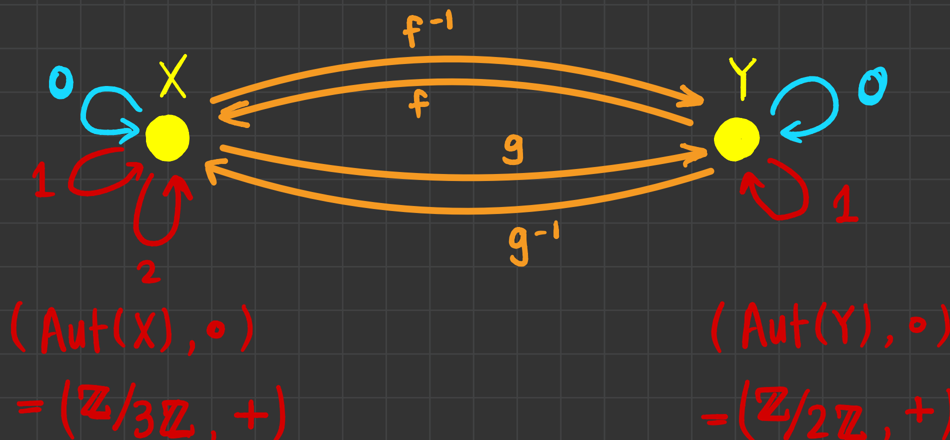
But if we want to track *how* objects are equivalent, then it's better to go back to:



Definition: A groupoid is a category where every morphism is an isomorphism.

Groupoids

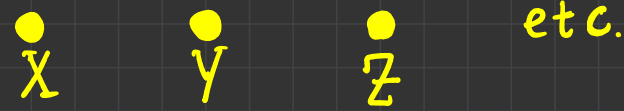
Groupoid = A bunch of objects/vertices. At each vertex, the arrows/morphisms form a group. But we allow the possibility move between vertices. All paths between vertices are “two way streets”.



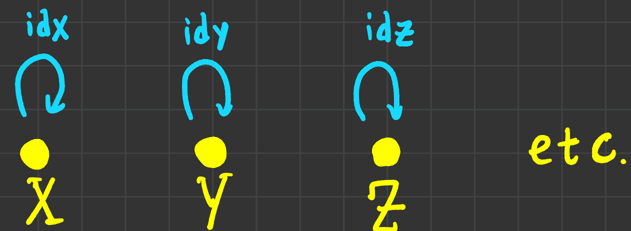
Example: Sets

Start with a set: $X = \{x, y, z, \cdots\}$

Make a groupoid with set elements as objects.



and to each set element, attach a trivial-group's-worth of arrows.



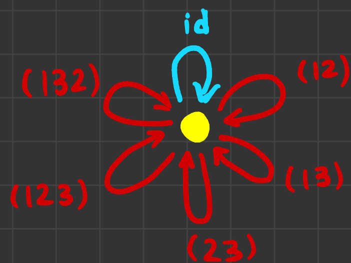
No way of traveling between vertices.

Example: Groups

Start with a group: $G = S_3$

Make a groupoid with a single object:

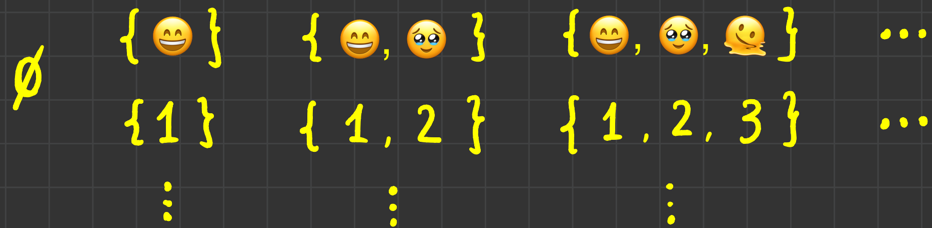
To that object, attach the group G as its arrows.



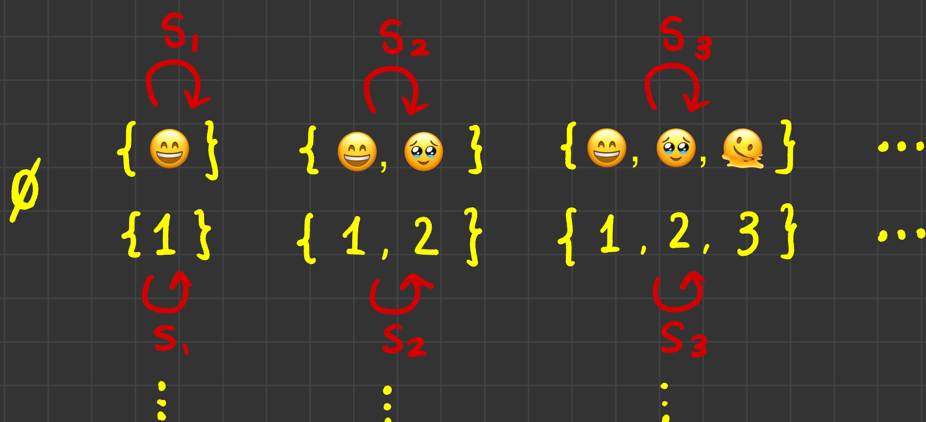
We call this category the classifying groupoid **B**G of G.

Example: Finite Sets

Start with a finite sets as objects:

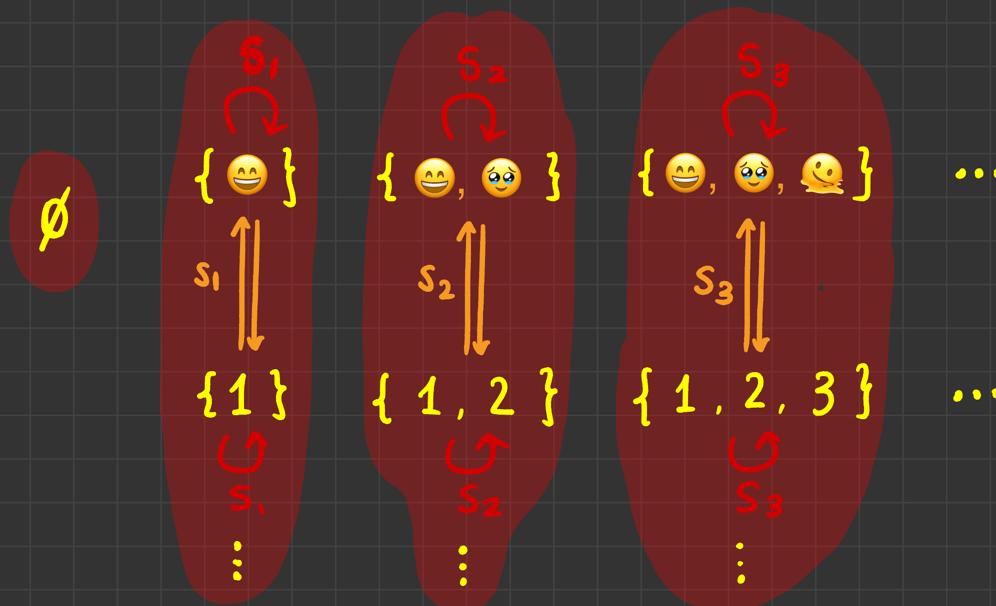


Each set has a symmetric group's number of automorphisms:



Among each pairs of sets of the same size, we also have a symmetric group of arrows.

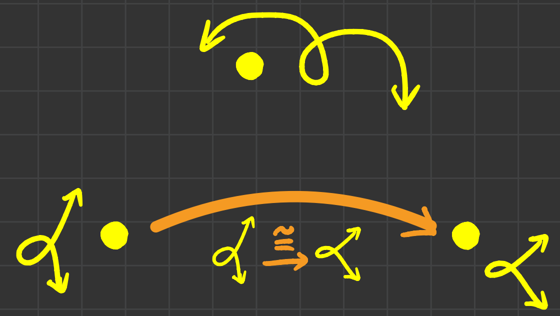
Among each pairs of sets of the same size, we also have a symmetric group of arrows.



Notice we have $\mathbb{N}$ many isomorphism classes / connected components.

Example: Smooth Curves

Take “smooth connected projective genus g curves over $\mathbb{C}$ ” as objects.

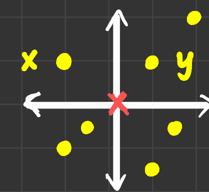


And make the maps between curves isomorphism of curves.

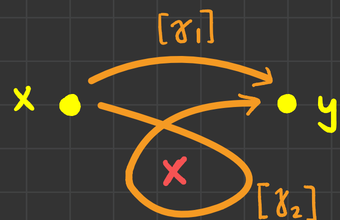
Example: Fundamental Groupoids

Let X be a topological space. e.g. $\mathbb{R}^2 - 0$

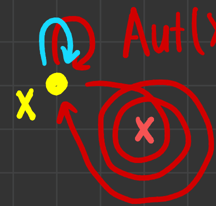
Choose the points in X to be your objects.



Add one arrow $x \xrightarrow{[\gamma]} y$ for each equivalence class of paths from x to y, modulo homotopy.



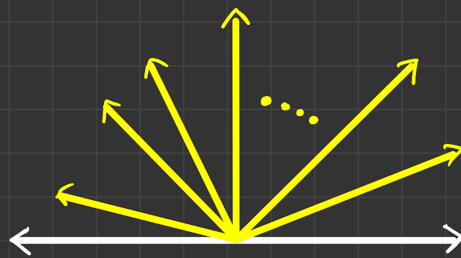
The automorphisms of each point x forms the fundamental group $\pi_1(X, x_0)$.

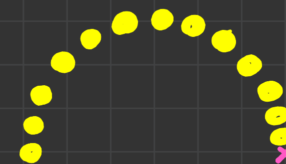


$$\mathrm{Aut}(X) = \pi_1(X, x_0) = \mathbb{Z}$$

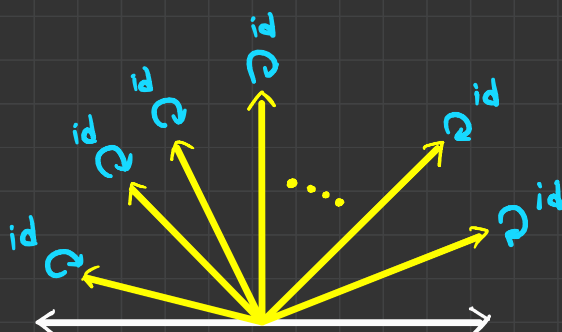
Example: Projective Space 1

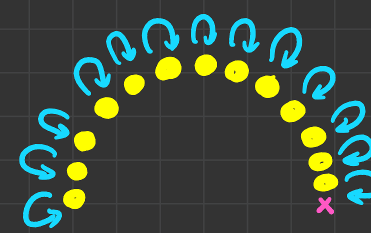
Start with lines in $\mathbb{R}^{n+1}$ as objects:





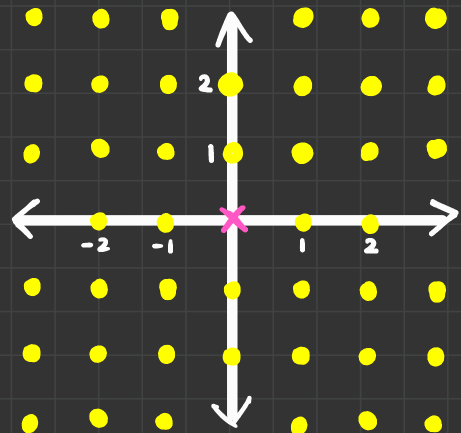
To each line, add a trivial group as arrows.





Example: Projective Space 2

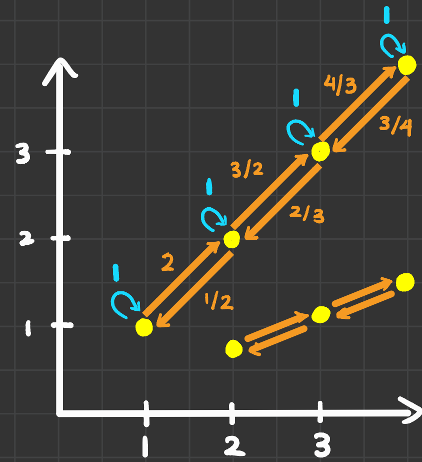
Start with points in the punctured plane $\mathbb{R}^{n+1} - \{\vec{0}\}$ as objects.



Example: Projective Space 2

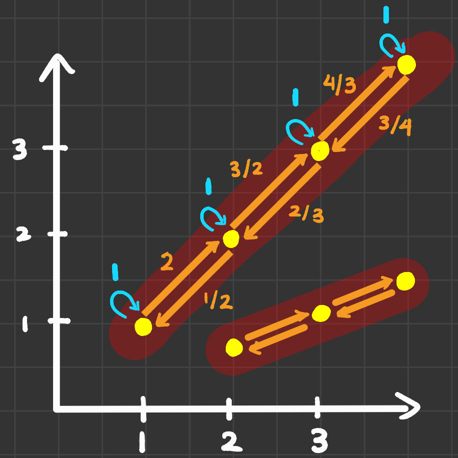
For each pair of points $\vec{x}, \vec{y} \in \mathbb{R}^{n+1} - 0$, add a unique arrow $\vec{x} \xrightarrow{\lambda} \vec{y}$

if $\vec{y} = \lambda \vec{x}$.



Example: Projective Space 2

The isomorphism classes form the “lines”.

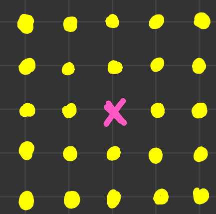


Example: Moduli Groupoid of Orbits

Let G be a group acting on a set X. $G \curvearrowright X$

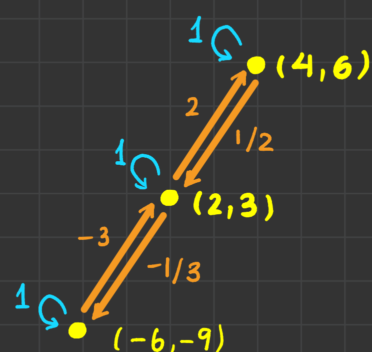
Make the elements of X your objects.

$X = \mathbb{R}^{n+1} - \{\vec{0}\}$



Add an arrow $x \xrightarrow{g} y$ if $g \in G$ and $y = g \cdot x$.

$G = (\mathbb{R}^{\times}, \times)$



Notice $\forall x \in X$:

$$\mathrm{Aut}(x) = \underbrace{\{g \in G \mid g \cdot x = x\}}_{\text{stabilizer subgp of } x}$$

stabilizer subgp of x

Equivalent Groupoids

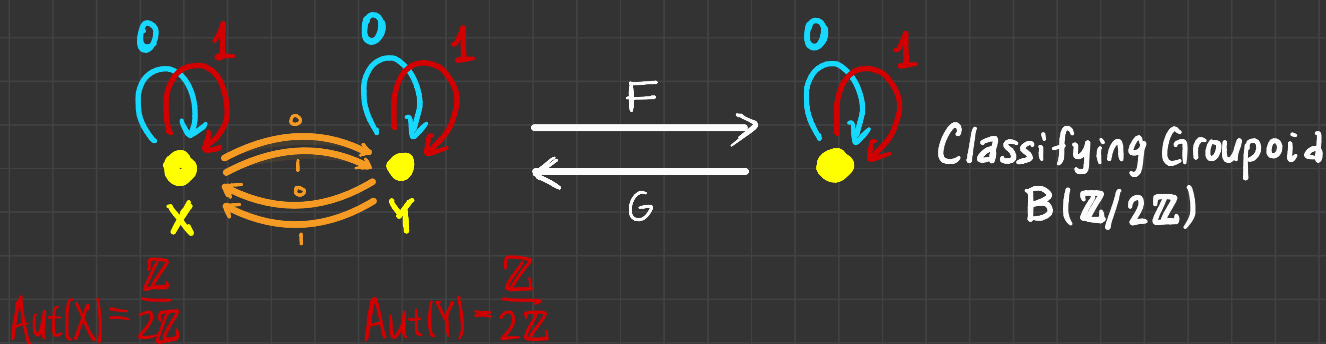
Definition: Two groupoids C, D are equivalent if there's a functor $C \xrightarrow{F} D$ between them such that:

(1) the assignment on objects hits every isomorphism class

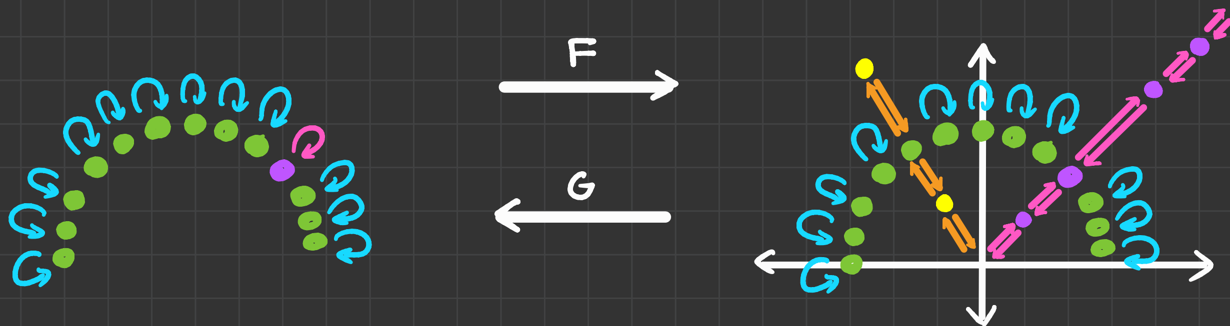
$\forall Y \in D : \exists X \in C : FX \cong Y$

(2) for every fixed pair of objects, the assignment on morphisms is bijective.

$\forall X, Y \in C : \{X \to Y\} \overset{1-1}{\longleftrightarrow} \{FX \to FY\}$



Example: Projective Space 1 & 2 are equivalent

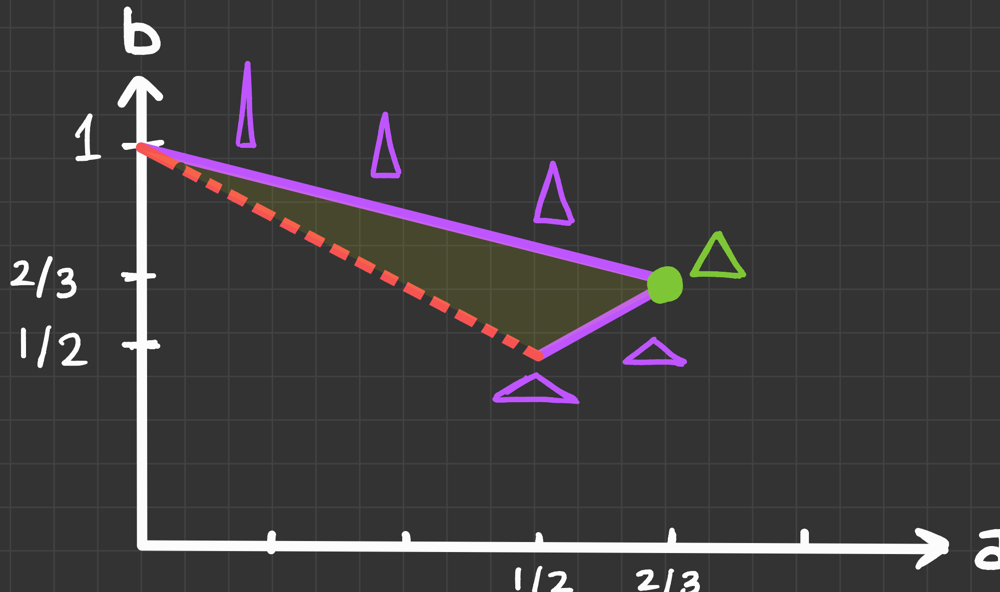


Example: Unlabelled Triangles

$M^{unl} :=$ Moduli set of triangles in $\mathbb{R}^2$, with unlabelled sides, modulo similarity.

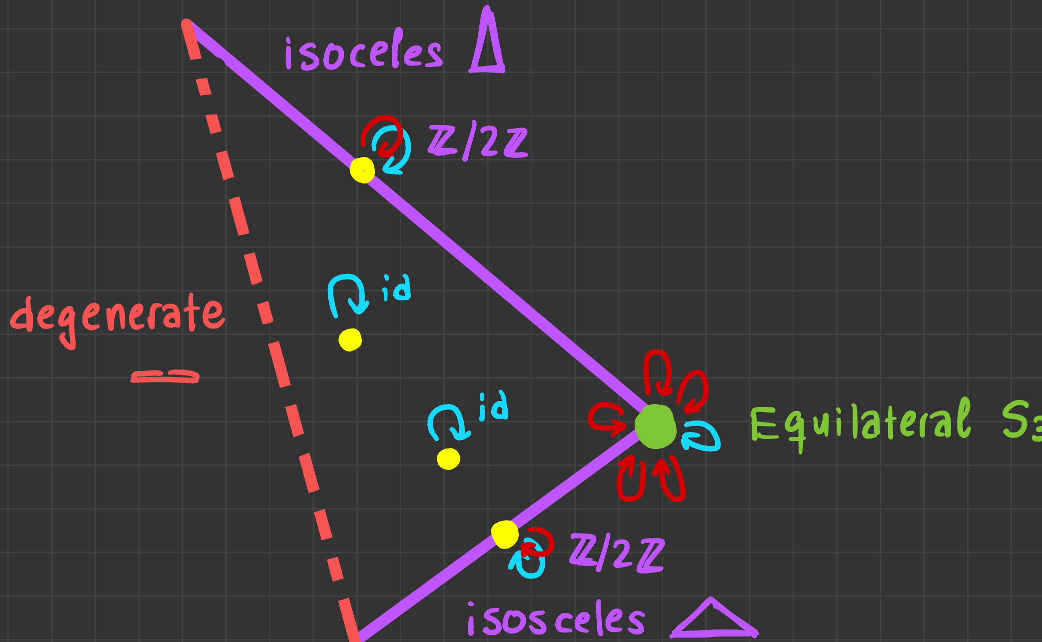
$$M^{unl} = \{ (a,b,c) \mid 0 < a \le b \le c < a+b \text{ and } a+b+c = 2 \}$$

- (a,b,c): triples of side lengths
- $a \le b \le c$: ordered small to big
- $0 < a$ and $c < a+b$: and avoiding the degenerate cases
- $a+b+c = 2$: choose representative in equivalence class with perimeter 2



b
1
2/3
1/2
a
1/2
2/3

We can artificially "add the iso/symmetries back in".



We get a moduli groupoid $\mathcal{M}^{unl}$.

Example: Labelled Triangles

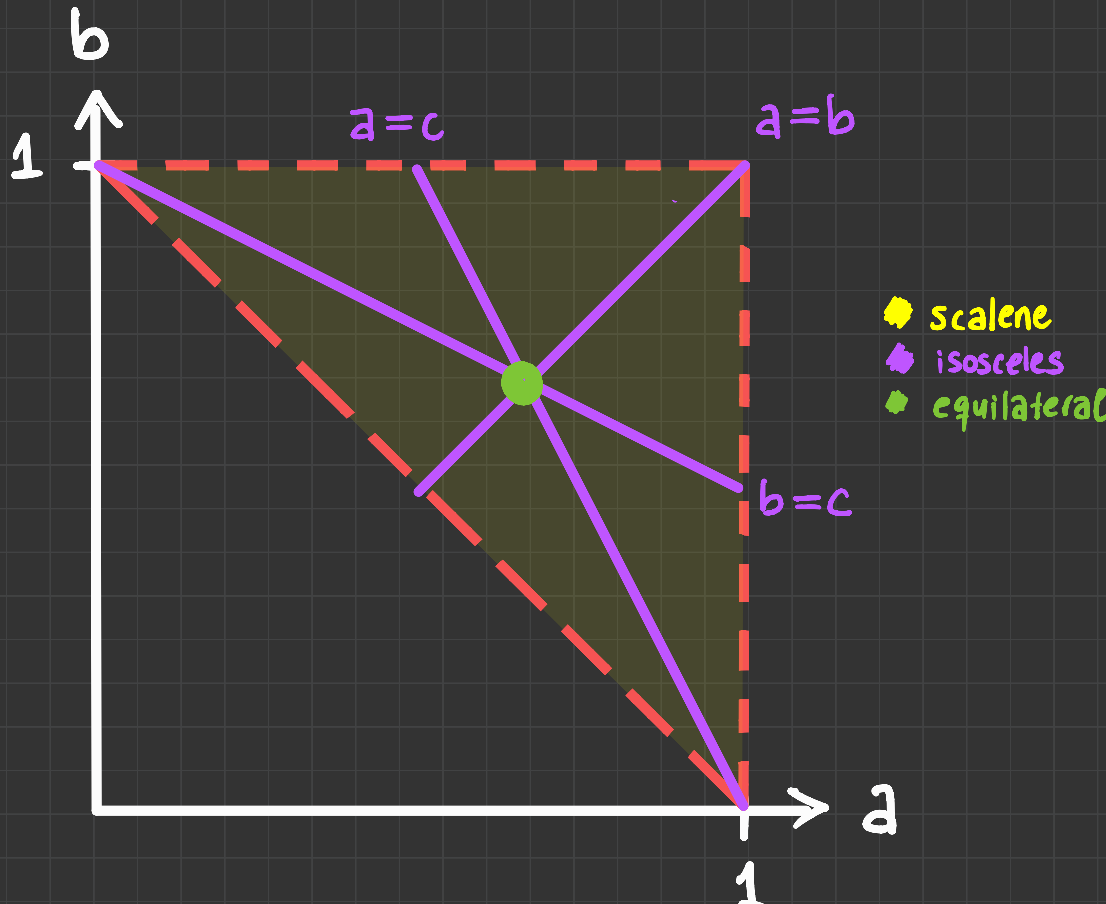
M^{lab} = triangles with colored sides, modulo similarity that matches colors.

$$= \left\{ (a, b, c) \;\middle|\; \begin{array}{l} 0 < a < b+c \\ 0 < b < a+c \\ 0 < c < a+b \end{array} \text{ and } \quad a+b+c=2 \right\}$$

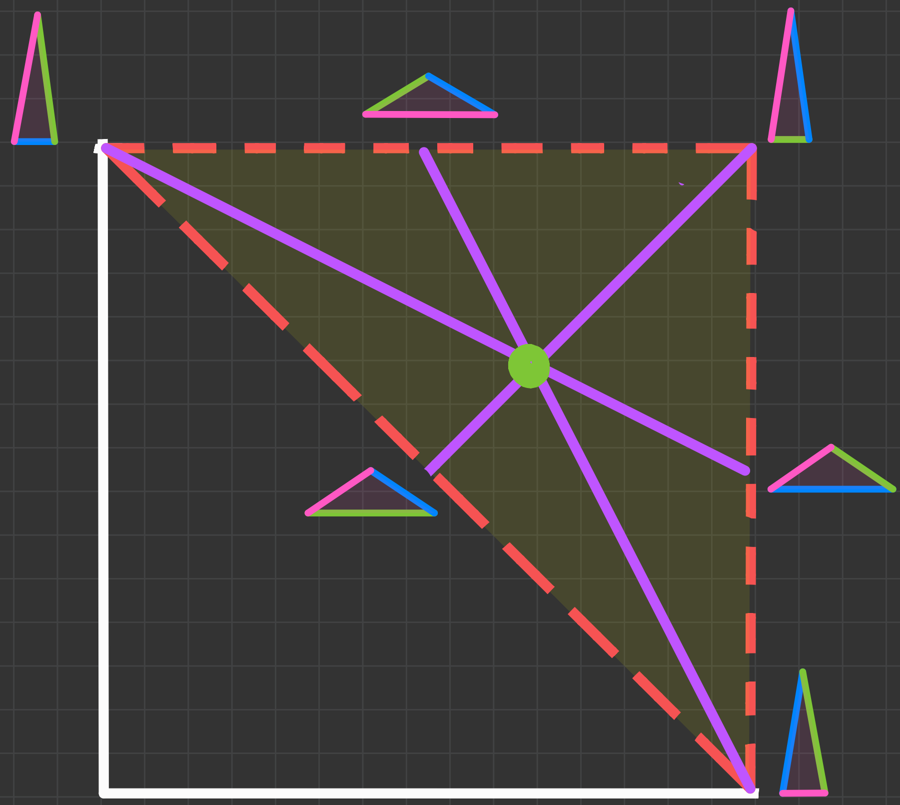
↑ triples of side lengths

↑ side lengths positive and ensure triangle closes up

↑ equiv. class representative chosen with perimeter 2.



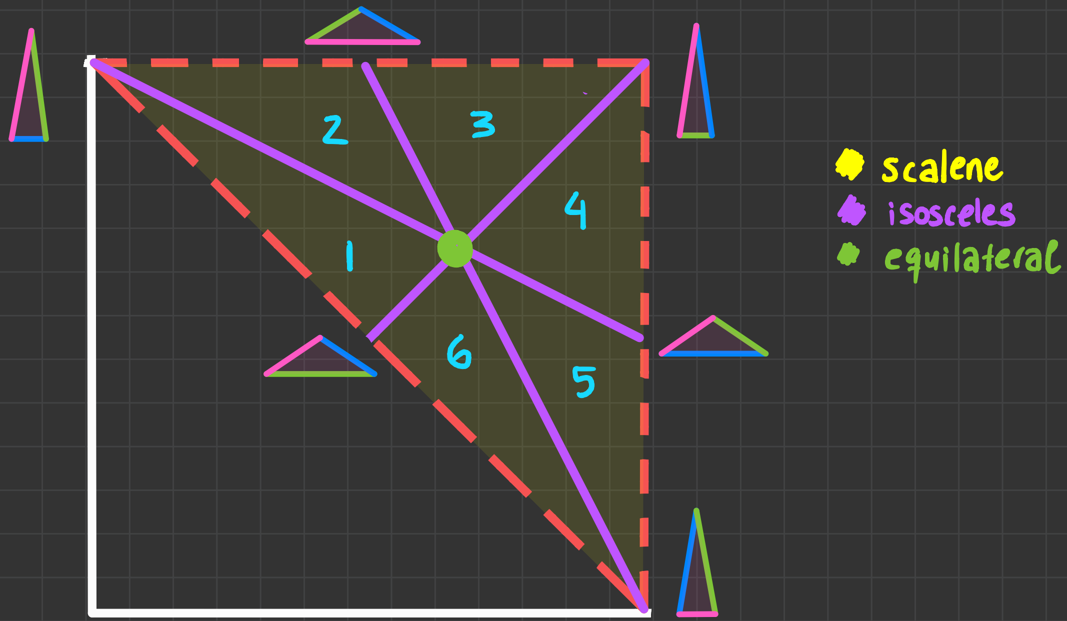
b
1
a=c
a=b
b=c
a
1
scalene
isosceles
equilateral



Again we can add isomorphisms to each point, based on the triangles symmetry.

The corresponding moduli groupoid $\mathcal{M}^{lab}$ has no way of traveling between different vertices.

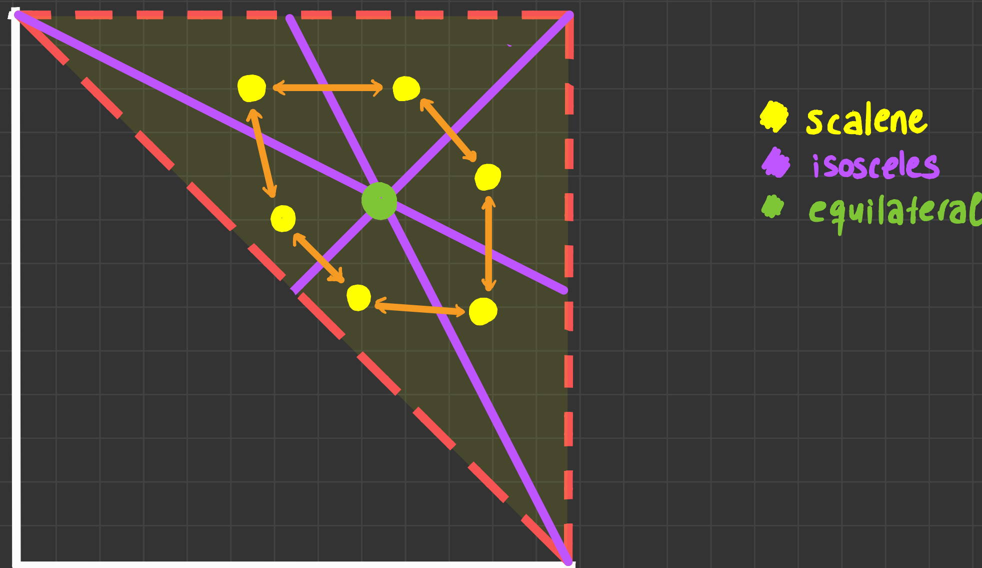
We can construct a moduli groupoid differently.

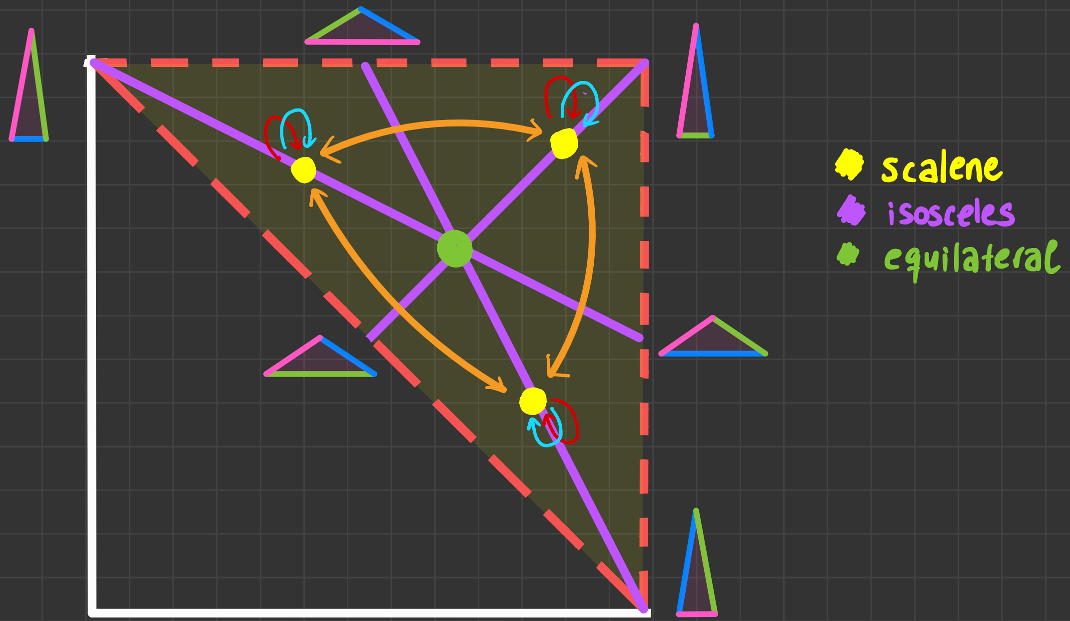


The six portions each contain one triangle for each ratio a:b:c of side lengths.

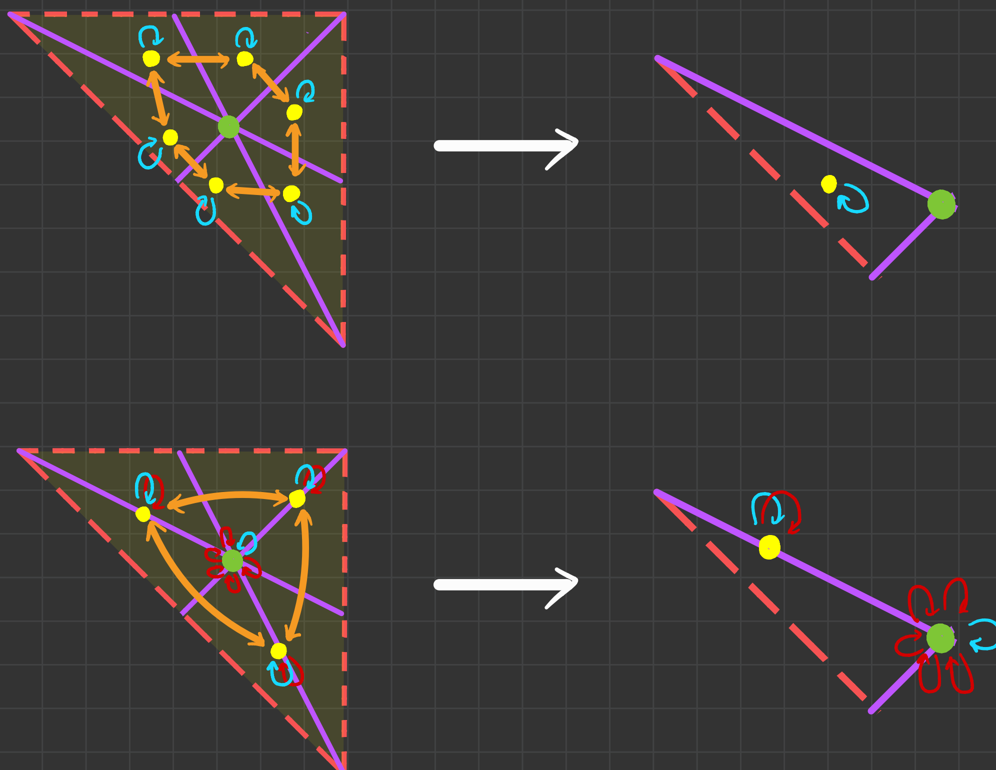
The only reason why the six triangles with ratio a:b:c are different is because the colors of the sides don't match.

But we can define an S action on M^{lab} by permuting the colors of the sides.




scalene
isosceles
equilateral

This new moduli groupoid M^{lab}/S_3 is equivalent to $\mathcal{M}^{unl}$.

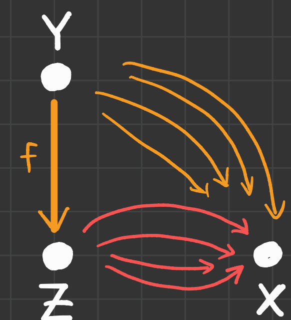


Yoneda Lemma

For a category $\mathcal{C}$ with object $X \in \mathcal{C}$, the functor h_X represented by X is:

$$h_X : \mathcal{C} \longrightarrow \mathrm{Set}$$

$$\begin{array}{ccc} Y & & \mathrm{Hom}(Y,X) \\ f \downarrow & \longmapsto & \uparrow \\ Z & & \mathrm{Hom}(Z,X) \end{array}$$



Yoneda Lemma

For $X \in \mathcal{C}$ and $\mathcal{C} \xrightarrow{F} \mathrm{Set}$:

$$\mathrm{Hom}(h_X, F) \overset{\cong}{\longleftrightarrow} FX$$

"representations" of the presheaf F

are fully determined by

where a single element is sent

Yoneda Embedding

$$\mathcal{C} \hookrightarrow \mathrm{Functors}(\mathcal{C}^{op} \to \mathrm{Set})$$

$$\begin{array}{ccc} X & & h_X \\ f \big\downarrow & \rightsquigarrow & \big\downarrow f_* \\ Y & & h_Y \end{array}$$

Lightning Example

We can recover a variety X over $\mathbb{C}$ from its functor of points.

Why is Yoneda useful?

$$\mathcal{C} \hookrightarrow \text{Functors}(\mathcal{C}^{op} \to \text{Set})$$

Allows us to naturally embiggen our category and move to a new setting with more arrows.

More arrows = more freedom to get places.

Two roads diverged in the woods…

Moral : but actually, I found a third road after applying the Yoneda embedding…

but actually, it’s not really a “road”, it’s more like a “trail”?

What's the issue?

Once we abandon $\mathcal{C}$ and move to $\text{Functors}[\mathcal{C}^{op} \to \text{Set}]$, we can't "go back" unless we know that the object/functor we are at is in the "image" of the Yoneda embedding.

Definition

A functor $\mathcal{C} \xrightarrow{F} \text{Set}$ is representable if it's in the "image" of the Yoneda embedding.

i.e. $\exists X \in \mathcal{C} : \quad F \cong h_X .$

i.e. $\exists X \in C$ and natural isos. $\{FY \xrightarrow{\cong} \text{Hom}(Y, X)\}_{Y \in C}$

$\{FY \xrightarrow{\cong} \text{Hom}(X, Y)\}_{Y \in C}$

Example

$$\text{Set} \xhookrightarrow{\text{Yoneda}} \text{Functors}(\text{Set} \to \text{Set})$$

$$\rotatebox{90}{$\in$}$$

$$\boxed{?} \rightsquigarrow \text{Set} \xrightarrow{id} \text{Set}$$

$$\begin{array}{c} X \\ f\downarrow \\ Y \end{array} \mapsto \begin{array}{c} X \\ f\downarrow \\ Y \end{array}$$

$F = id$ is representable if...

$\exists\ \boxed{?} \in \text{Set}$ and nat. isos : $X \xrightarrow{\cong} \text{Hom}(\boxed{?}, X)$

Representable! $\boxed{?} = \{*\}$ one element set

Example

$$\mathbf{Top} \xhookrightarrow{\text{Yoneda}} \text{Functors}(\mathbf{Top}^{op} \to \mathbf{Set})$$

$$\rotatebox{90}{$\in$}$$

[?] ⇝ open sets functor $\mathcal{O}$

$$\mathbf{Top} \longrightarrow \mathbf{Set}$$

$$(X, \tau) \mapsto \tau$$

$$f \downarrow \quad \rightsquigarrow \quad \uparrow f^{-1}$$

$$(Y, \tau') \mapsto \tau'$$

$\mathcal{O}$ is representable if...

$\exists$ [?] $\in \mathbf{Top}$ and nat. isos : $\tau \xrightarrow{\cong}$ [continuous maps $X \to$ [?]]

Representable! [?] = ● ○ (closed point, open point) ← also represents the closed sets functor

Example

$$\mathrm{Vect}_{\mathbb{k}} \overset{\text{yoneda}}{\hookrightarrow} \text{Functors}(\mathrm{Vect}_{\mathbb{k}}^{op} \to \mathrm{Set})$$

$$\boxed{?} \rightsquigarrow \text{dual space functor} \in \text{Functors}(\mathrm{Vect}_{\mathbb{k}}^{op} \to \mathrm{Set})$$

$$\mathrm{Vect}_{\mathbb{k}} \xrightarrow{(-)^*} \mathrm{Set}$$

$$\begin{array}{ccc} V & & V^* \\ T \downarrow & \rightsquigarrow & \uparrow T^* \\ W & & W^* \end{array}$$

$(-)^*$ is representable if...

$\exists \ \boxed{?} \in \mathrm{Vect}_{\mathbb{k}}$ and nat. isos : $V^* \overset{1-1}{\leftrightarrow} [\text{Linear Transfs } V \to \boxed{?}\,]$

Representable by $\mathbb{k}$!

Example The functor $(X, \mathcal{O}_X) \to \{ (f_1, \cdots, f_n) \mid f_i \in \Gamma(X, \mathcal{O}_X) \}$ is represented by affine n-space $\mathbb{A}^n$.

For affine schemes $\mathrm{Aff} \equiv \mathrm{CRing}^{op}$ ...

The functor $\mathrm{CRing} \xrightarrow{F} \mathrm{Set}$ $\quad R \to R^n$ is representable:

$$\left[\text{Ring Homs } \boxed{?} \to R \right] \overset{1-1}{\longleftrightarrow} R^n$$

$\uparrow$

$\mathbb{Z}[x_1, \cdots, x_n]$

Example The functor $(X, \mathcal{O}_X) \to$ invertible fns. on X is represented by $\operatorname{Spec} \mathbb{Z}[x, x^{-1}]$.

For affine schemes $\mathrm{Aff} \equiv \mathrm{CRing}^{op}$ ...

The functor $\mathrm{CRing} \xrightarrow{F} \mathrm{Set}$ $\quad R \to R^\times$ is representable:

$$\left[\text{Ring Homs } \boxed{?} \to R\right] \overset{1-1}{\longleftrightarrow} R^\times$$

$\uparrow$

$\mathbb{Z}[x, x^{-1}]$