PRIMER ON REPRESENTATION THEORY

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1. Basic Notions

1.1. Representations of algebras.

Definition 1. Let k be a field, and A a k-algebra. A **representation** of A consists of a k-vector space V and a algebra homomorphism $\rho : A \to \mathrm{End}_k(V)$. We denote the image of $a \in A$ by ρ_a and write $a \cdot v := \rho_a(v)$ for $v \in V$ to suggest each element of $a \in A$ acts on V by some endomorphism.

A **homomorphism of representations** $(V, \rho^V) \to (W, \rho^W)$ is a linear map $\phi : V \to W$ satisfying $\phi(a \cdot v) = a \cdot \phi(v)$ for all $a \in A, v \in V$. We denote by Rep_A the category of k-representations of A.

The **dimension** of a representation is simply the dimension of V as a vector space.

Remark (Representations=Modules). Using the tensor-hom adjunction of k-vector spaces, the data of a representation $\rho : A \to \mathrm{End}_k(V) = \mathrm{Hom}_k(V, V)$ is equivalent to the following k-linear map $A \otimes V \to V$,

$$A \otimes V \xrightarrow{\rho \otimes 1} \mathrm{End}(V) \otimes V \xrightarrow{ev} V \qquad a \otimes v \mapsto \rho_a \otimes v \mapsto a \cdot v = \rho_a(v).$$

Tracing through this correspondence, the requirement that ρ be an algebra map (not just a k-linear map) correspond precisely to the A-module axioms:

$$\rho(ab) = \rho(a)\rho(b) \qquad \Leftrightarrow \qquad ab \cdot v = a \cdot (b \cdot v),$$

$$\rho(1_A) = 1_V \qquad \Leftrightarrow \qquad 1_A \cdot v = v.$$

Similarly, homomorphisms of representations correspond to A-module homomorphisms. In fact, Rep_A is isomorphic to the category of left A-modules $\mathbf{Mod}_A$. This means the study of representations and modules is one and the same, and formalizes the notation $a \cdot v = \rho_a(v)$.

Definition 2. A **subrepresentation** of (V, ρ) is a subspace $W \subseteq V$ invariant under all A-actions, i.e. $a \cdot W \subseteq W$ for all $a \in A$. In $\mathbf{Mod}_A$, these correspond to **submodules** of V.

Definition 3. An **irreducible representation** is one with no nontrivial proper subrepresentations. In $\mathbf{Mod}_A$, these correspond to the **simple** A-modules.

Definition 4. Given representations V_1, V_2 of A, their **direct sum** $V_1 \oplus V_2$ is a representation via $a \cdot (v_1 \oplus v_2) = a \cdot v_1 \oplus a \cdot v_2$. A nonzero representation is **indecomposable** if it's not isomorphic to a direct sum of two nonzero representations.

Definition 5. A representation is **completely reducible** if it is a direct sum of (its) irreducible (sub)representations. In $\mathbf{Mod}_A$, these modules are called **semisimple**, and they decompose as (internal) direct sums of (their own) simple (sub)modules.

1.2. **Representations of groups.**

Definition 6. A **group representation** for a group G over k consists of a k-vector space V along with a group homomorphism $G \to \mathrm{End}_k(V)$.

Remark (Representations=Linear Actions). Most of us encounter **group actions** before we encounter representations. These consist of a group G, a set X, and a function $G \times X \to X$ satisfying:

$$\forall g, h \in G, x \in X: \quad gh \cdot x = g \cdot (h \cdot x) \quad \text{and} \quad 1_G \cdot x = x.$$

This definition is eerily similar to that of a module. In fact, using the tensor-hom adjunction in the category of **Set** (aka currying $\mathrm{Hom}_{\mathbf{Set}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathbf{Set}}(X, \mathrm{Hom}_{\mathbf{Set}}(Y, Z))$), a group action is equivalent to a group homomorphism $G \to S_X = \mathrm{End}_{\mathbf{Set}}(X)$ to the symmetric group on X. We should also note that we don't need G above to be a group. Any object with an associative binary operation and identity element (i.e. any monoid) can fit the definition.

From this vantage point, we may think of group/monoid actions as set-theoretic versions of representations. Alternatively, representations are the k-linear analogs of group actions.

Definition 7. Given a group G and a ring R, the group ring $(R[G], +, \cdot)$ has:

- $(R[G], +)$ is the free R-module with basis G.
- $(R[G], \cdot)$ extends the group law on the basis R-linearly, e.g. $rg \cdot \sum_i r_i g_i = \sum_i (rr_i)(gg_i)$.

When R is commutative, the R-scalars can move around freely, so the multiplication map $R[G] \times R[G] \to R[G]$ becomes R-bilinear, and $R[G]$ becomes an R-algebra.

As in the remark above, group algebras $k[G]$ are simply k-linear analogs of groups.

Remark (Group Reps are Reps). Group representations are instances of algebra representations. Using the tensor-hom adjunction, notice that group representations are equivalent to modules over the group algebra $k[G]$.

$$k[G] \otimes V \to V \quad \rightsquigarrow \quad k[G] \to \mathrm{Hom}(V, V) = \mathrm{End}(V).$$

Remark. Categorically, the group ring functor $R[-] : \mathbf{Group} \to \mathbf{Alg}_R$ is left adjoint to the group of units functor $(-)^\times : \mathbf{Alg}_R \to \mathbf{Group}$. The universal property of $R[G]$ states that group homomorphisms $f : G \to S^\times$ to R-algebras S are in correspondence with R-algebra homomorphisms $\bar{f} : R[G] \to S$ making the following commute:

$$\begin{array}{ccc} G & \hookrightarrow & R[G] \\ & {\scriptstyle f} \searrow & \downarrow {\scriptstyle \bar{f}} \\ & & S \end{array}$$

Remark (Group rings as Convolution Algebras). It is also helpful to define the group ring $R[G]$ as the collection of functions $G \to R$ with finite support. Addition and R-scaling work as expected, while the product of two functions$f, f' : G \to R$ is given by convolution:

$$(f \cdot f')(g) := \sum_{hh'=g} f(h)f'(h') = \sum_{h \in G} f(h)f'(h^{-1}g).$$

In particular, when G is finite we may identify $R[G]$ as the space of R-valued functions on G.

1.3. Representations of Lie algebras.

Definition 8. A k-vector space $\mathfrak{g}$ is a **Lie algebra** if it comes equipped with a skew-symmetric bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity:

- (Skew-Symmetry) $\forall x, y \in \mathfrak{g}$: $[x, x] = 0$ and $[x, y] = -[y, x]$.
- (Jacobi Identity) $\forall x, y, z \in \mathfrak{g}$ we have $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$.

A **homomorphism** of Lie algebras $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a linear map preserving the brackets:

$$\varphi([a,b]) = [\varphi(a), \varphi(b)].$$

Example 9. Any algebra A is a Lie algebra via the commutator bracket $[a,b] := ab - ba$. In particular, for any $V \in \mathbf{Vect}$, $\mathrm{End}_k(V)$ is a Lie algebra, which we denote $\mathfrak{gl}(V)$.

Definition 10. Let $\mathfrak{g}$ be a Lie algebra. The **universal enveloping algebra** $\mathcal{U}(\mathfrak{g})$ is the quotient $\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/\sim$ of the tensor algebra $T(\mathfrak{g})$ by the relations $a \otimes b - b \otimes a = [a,b]$. As such, it comes equipped with a canonical map $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$.

Recalling that algebras can always be considered Lie algebras via the commutator bracket, the universal property of $\mathcal{U}(\mathfrak{g})$ gives a correspondence between Lie algebra maps from $\mathfrak{g}$ and algebra maps from $\mathcal{U}(\mathfrak{g})$. Namely, for any algebra A, Lie algebra maps $\varphi : \mathfrak{g} \to A$ are in correspondence with algebra homomorphisms $\hat{\varphi} : \mathcal{U}(\mathfrak{g}) \to A$ satisfying $\varphi = \hat{\varphi} \circ i$.

If $\mathfrak{g}$ has a basis x_i and relations $[x_i, x_j] = \sum_k c_{i,j}^k x_k$, then $\mathcal{U}(\mathfrak{g})$ is generated by x_i with defining relations $x_i x_j - x_j x_i = \sum_k c_{i,j}^k x_k$.

Example 11. For any algebra A, a linear map $D : A \to A$ is a **derivation** if it satisfies Leibniz's rule:

$$D(ab) = D(a)b + aD(b).$$

The space of derivations $\mathrm{Der}(A)$ is a Lie algebra with commutator bracket.

Definition 12. A **representation** of a Lie algebra $\mathfrak{g}$ is a vector space V along with a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V) = \mathrm{End}_k(V)$. Using the universal property of $\mathcal{U}(\mathfrak{g})$, we see that representations of Lie algebras are equivalent to representations of their universal enveloping algebras.

Remark (Jacobi=Leibniz). Every Lie algebra $\mathfrak{g}$ admits an **adjoint representation**

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \qquad x \mapsto \mathrm{ad}_x = [x, -].$$

The Jacobi identity states that the adjoint actions acts like derivations on $(\mathfrak{g}, [-,-])$:

$$\text{(Jacobi Identity)} \qquad \mathrm{ad}_x([y,z]) = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)].$$

Remark (Lie Algebras are Infinitesimal Automorphisms). Lie algebras arise as spaces of infinitesimal automorphisms (derivations) of algebras. Let A be a finite dimensional algebra over $\mathbb{R}$ or $\mathbb{C}$. Given a 1-parameter family of differentiable automorphisms $\{g(t) \in \mathrm{Aut}(A) \mid t \in (-\epsilon, \epsilon)\}$ with $g(0) = \mathrm{id}$, then $g'(0)$ is a derivation. Conversely, given a derivation, e^{tD} is a 1-parameter family of automorphisms.

Example 13. The space $\mathfrak{so}(n)$ of skew-symmetric $n \times n$ matrices is a Lie algebra using the commutator. For $n = 3$, $\mathfrak{so}(3)$ is isomorphic to $\mathbb{R}^3$ with the cross product.

The space $\mathfrak{sl}(n)$ of $n \times n$ trace zero matrices is a Lie algebra via the commutator. For $n = 3$, we have basis $e = (\delta_{1,2})$, $f = (\delta_{2,1})$, and $h = \mathrm{diag}(1, -1)$ and relations:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

The Heisenberg Lie algebra $\mathcal{H}$ of strictly upper triangular 3×3 matrices has basis $x = (\delta_{2,3})$, $y = (\delta_{1,2})$, and $c = (\delta_{1,3})$ with relations $[y, x] = c$ and $[y, c] = [x, c] = 0$.

1.4. General Results.

Theorem 14 (Schur's Lemma). Let $\phi : V \to W$ be a nonzero hom of representations.

- If V is irreducible, then ϕ is 1-1 because $\ker(\phi)$ is a subrepresentation.
- If W is irreducible, then ϕ is onto because $\mathrm{im}(\phi)$ is a subrepresentation.
- In particular, homs between irreducible representations are isomorphisms or zero.

Now let V be a finite-dimensional irreducible representation of A and $\phi : V \to V$ a hom.

- If k is algebraically closed, then $\phi = \lambda\,\mathrm{id}$ for some $\lambda \in k$.
- In particular, if A is commutative, irreducible reps are 1-dimensional (and vice versa).

Proof. For the next-to-last part, the characteristic polynomial of ϕ has a root λ, so $\phi - \lambda\,\mathrm{id}$ is a hom of irreducible representations that is *not* an isomorphism ($\det = 0$), so $\phi - \lambda\,\mathrm{id} = 0$.

For the last part, commutativity implies each ρ_g is a hom of representations, hence a scalar operator. So every subspace is a subrepresentation, which forces $\dim V = 1$ to avoid nontrivial subspaces. □

Theorem 15 (Subrepresentations of Semisimple Representations). Let V_i be irreducible finite dimensional pairwise nonisomorphic representations of A, and n_i be positive integers. The subrepresentations W of the semisimple $\bigoplus_i n_i V_i$ are of the form $\bigoplus_i r_i V_i$ with $r_i \leq n_i$.

Furthermore, the inclusion $\phi : W \hookrightarrow V$ is a direct sum of inclusions $\phi_i : r_i V_i \to n_i V_i$.

Theorem 16 (Density Theorem). Let k be algebraically closed and V be an irreducible finite dimensional representation of A.

(1) If $v_1, \ldots, v_n \in V$ are linearly independent, then $\forall w_1, \ldots, w_n \in V, \exists a \in A : av_i = w_i$.
(2) The map $\rho : A \to \mathrm{End}(V)$ is surjective.
(3) If (V_i, ρ_i) are irreducible pairwise nonisomorphic finite dimensional representations of A then the map $\bigoplus_i \rho_i : A \to \bigoplus_i \mathrm{End}(V_i)$ is surjective.

Theorem 17 (Representations of Matrix Algebras). Let $A = \bigoplus_{i=1}^r \mathrm{Mat}_{d_i}(\Bbbk)$. The irreducible representations of A are $k^{d_1}, k^{d_2}, \ldots, k^{d_r}$. Any finite dimensional representaion of A is a direct sum of copies of these.

Theorem 18 (Maschke's Theorem). Let G be a finite group and $\mathrm{char}(k) \nmid |G|$. Then:

(1) $k[G]$ is semisimple,
(2) $k[G] \cong \bigoplus_i \mathrm{GL}(V_i)$ as algebras, where V_i are the irreducible representations of G.
(3) The regular representation of $k[G]$ decomposes into $\bigoplus_i \dim(V_i) V_i$, giving the formula:

$$|G| = \sum_i \dim(V_i)^2.$$

2. Characters

Let A be an algebra and V a finite dimensional representation.

Definition 19. The **character** of V is the linear function $\chi : A \to \Bbbk$ given by:

$$\chi(a) = \mathrm{Tr}(\rho_a).$$

Remark. Since $\mathrm{Tr}(M+N) = \mathrm{Tr}(M) + \mathrm{Tr}(N)$ and $\mathrm{Tr}(MN) = \mathrm{Tr}(NM)$ for matrices M, N, it follows that $[A, A] \subseteq \ker(\chi)$. Using the universal property of the quotient, we may thus view the character of V as a map $\chi : A/[A, A] \to \Bbbk$ instead.

Theorem 20 (Independence of Characters). The characters of distinct irreducible finite dimensional representations of A are linearly independent. If A is a finite dimensional semisimple algebra, then these characters form a basis for $(A/[A, A])^*$.

Remark (Characters of Group Representations). Let V be a finite dimensional group representation of a finite group G. We define the **character** of V as $\chi : G \to \Bbbk$ with $\chi(g) = \operatorname{Tr}(\rho_g)$. This is simply the restriction of the usual character to the basis G of $\Bbbk[G]$, and carries the same information.

Group characters are **class functions**, i.e. they are constant on conjugacy classes (or equivalently, if $\chi(hg) = \chi(gh)$ for all $g, h \in G$). This follows since traces are constant under cyclic permutations:

$$\chi(hgh^{-1}) = \operatorname{Tr}(\rho_h \rho_g \rho_{h^{-1}}) = \operatorname{Tr}(\rho_g \rho_{h^{-1}} \rho_h) = \operatorname{Tr}(\rho_g) = \chi(g).$$

Viewing $\Bbbk[G]$ as the space of k-valued functions, we denote by $Z \subseteq \Bbbk[G]$ the space of k-valued class functions on G.

Theorem 21 (Character Basis for Finite Groups). Let G be a finite group with $char \Bbbk \nmid |G|$. The characters of the irreducible representations of G form a basis for the space of k-valued class functions Z. In particular, the number of irreducible representations of G equal the number of conjugacy classes in G. In characteristic zero, representations of G are uniquely determined by their characters, i.e. $\chi_V = \chi_W$ iff $V \cong W$.

Proof. By Maschke's Theorem, $A = \Bbbk[G]$ is semisimple and the characters form a basis for $(A/[A, A])^*$. But as vector spaces:

$$\begin{aligned}
(A/[A, A])^* &= \{\text{linear } \bar{f} : A/[A, A] \to k\} \\
&\cong \{\text{linear } f : A \to k \mid [A, A] \subseteq \ker(f)\} \\
&\cong \{\text{linear } f : \Bbbk[G] \to k \mid \forall g, h \in G : f(gh - hg) = 0\} \\
&= \{\text{linear } f : \Bbbk[G] \to k \mid \forall g, h \in G : f(gh) = f(hg)\} = Z.
\end{aligned}$$

□

Example 22 (Dual Group of Finite Abelian Group). Let $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$. Denote by $G^\vee$ the set of irreducible representations of G over $\mathbb{C}$. Abelian implies every element is in its own conjugacy class, so $|G^\vee| = |G|$. Since $\mathbb{C}$ is algebraically closed, the irreducible representations are 1-dimensional (Schur's Lemma), and look like $\rho : G \to \operatorname{Aut}(\mathbb{C}) = \mathbb{C}^\times$.

When $G = \mathbb{Z}_n$, by finiteness, irreducible representations $\rho : \mathbb{Z}_n \to \mathbb{C}^\times$ must map $1 \mapsto \omega_n$, and so $\mathbb{Z}_n^\vee = \{\rho^k \mid k = 0, 1, \ldots, n-1\}$

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