Math 211 - Written HW 4

due date: 10PM Monday 02/26/2024

Learning Objectives

1. **Computational Practice:** This week, we covered matrix inverses and determinants and how to compute them. Problems 1 and 2 will give you additional methods to compute these. Try them out and see which method suits your computational style. Additional computational practice problems are also available on Webwork.

2. **Conceptual Review and Mastery:** Last week, we covered vector spaces, subspaces, and linear combinations. Problems 3-5 review these concepts with a heavier focus on building your theoretical understanding and mathematical intuition.

3. **Practice with Abstract Notation and Communication:** Last week, we saw how set notation is used to write/encode vector spaces and subspaces. Problems 3 and 4 help you practice translating between (written) set notation and (spoken/speakable) "natural language".

4. **Value Understanding over Rote Computation:** Last week, we covered how to compute RREFs to answer questions about linear combinations and span. Problem 5 will explicitly ask you to forego these algorithmic methods and use your conceptual understanding instead. Remember: if algorithms are problems solving procedures, the fastest algorithm is the one that doesn't have to run at all!

Problem 1. [Basket weaving for 3×3 determinants] An alternative method for computing the determinant of a 3×3 matrix:

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

is as follows. First, copy the first two columns of your matrix and put them on the right of the matrix:

$$\left[\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right]\begin{matrix} a & b \\ d & e. \\ g & h \end{matrix}$$

Next, add the products of the entries along the "downward moving diagonals" shown below in blue ($aei + bfg + cdh$):

$$\left[\begin{matrix} \mathbf{a} & b & c \\ d & \mathbf{e} & f \\ g & h & \mathbf{i} \end{matrix}\right]\begin{matrix} a & b \\ d & e \\ g & h \end{matrix} \qquad \left[\begin{matrix} a & \mathbf{b} & c \\ d & e & \mathbf{f} \\ g & h & i \end{matrix}\right]\begin{matrix} a & b \\ d & e \\ \mathbf{g} & h \end{matrix} \qquad \left[\begin{matrix} a & b & \mathbf{c} \\ d & e & f \\ g & h & i \end{matrix}\right]\begin{matrix} a & b \\ \mathbf{d} & e \\ g & \mathbf{h} \end{matrix}$$

Substract from it the products along the "upward moving diagonals" ($-gec - hfa - idb$):

$$\left[\begin{matrix} a & b & \mathbf{c} \\ d & \mathbf{e} & f \\ \mathbf{g} & h & i \end{matrix}\right]\begin{matrix} a & b \\ d & e \\ g & h \end{matrix} \qquad \left[\begin{matrix} a & b & c \\ d & e & \mathbf{f} \\ g & \mathbf{h} & i \end{matrix}\right]\begin{matrix} \mathbf{a} & b \\ d & e \\ g & h \end{matrix} \qquad \left[\begin{matrix} a & b & c \\ d & e & f \\ g & h & \mathbf{i} \end{matrix}\right]\begin{matrix} a & \mathbf{b} \\ \mathbf{d} & e \\ g & h \end{matrix}$$

Ultimately you get $\det(\mathbf{A}) = aei + bfg + cdh - gec - hfa - idb$.

Use the method above to answer the following question without performing any row reduction. State all values of k which make the following matrix invertible:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & k \\ 0 & k & 1 \\ k & 0 & 9 \end{bmatrix}.$$

Definition. For an $n \times n$ matrix $\mathbf{A}$, recall that the (i,j)-th **minor** of $\mathbf{A}$ (denoted $A_{i,j}$) is the matrix obtained by deleting row i and column j. The (i,j)-th **cofactor** of $\mathbf{A}$ (denoted $c_{i,j}$) is the determinant of $A_{i,j}$ multiplied by the (i,j)-th entry of the sign matrix of $\mathbf{A}$. Namely:

$$c_{i,j} = (-1)^{i+j} \det(A_{i,j}).$$

So for example, the matrix:

$$\mathbf{B} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 2 & -5 \\ 3 & 1 & 1 \end{bmatrix} \quad \rightsquigarrow \quad \text{sign matrix} = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix},$$

the $(3,2)$ minor and cofactor of $\mathbf{B}$ are:

$$B_{3,2} = \begin{bmatrix} -1 & 0 \\ -2 & -5 \end{bmatrix} \quad \rightsquigarrow \quad c_{3,2} = (-1)\det(B_{3,2}) = -5.$$

Problem 2. [A formula for matrix inverses] Given an $n \times n$ matrix $\mathbf{A}$, the "matrix of cofactors" of $\mathbf{A}$ is the $n \times n$ matrix whose (i,j)-th entry is the cofactor $c_{i,j}$. We will denote the matrix of cofactors of of $\mathbf{A}$ by $\tilde{\mathbf{A}}$.

If $\mathbf{A}$ is invertible, then its inverse is given by the formula:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \tilde{\mathbf{A}}^{\mathsf{T}}.$$

Notice we are taking the *transpose* of the matrix of cofactors.

(a) Use the formula above to write down a general formula for the inverse of an invertible 2×2 matrix:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(b) Use the formula above to compute the inverse of the matrix:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}.$$

Problem 3. [Identifying vector subspaces] Each of the following sets (a)-(f) is a subset of a vector space we introduced last week.

(a) The set of vectors in $\mathbb{R}^2$ in the first quadrant.

(b) The set of polynomials of degree exactly 2.

(c) The set of polynomials of even degree.

(d) The set of 2×2 matrices of determinant equal to zero.

(e) The set of 2×2 invertible matrices.

(f) The set of continuous functions $f : [0, 1] \to \mathbb{R}$ such that $f(0) = 1$.

For each set above, please:

1. Identify a vector space that contains the set.

2. Rewrite the set using set notation. Here is an example with various possible solutions:

$$\begin{aligned}\text{“the set of vectors in } \mathbb{R}^2 \text{ with second entry zero”} &\leftrightsquigarrow \{(x, y) \in \mathbb{R}^2 \mid y = 0\} \\ &\leftrightsquigarrow \{(x, 0) \in \mathbb{R}^2\} \\ &\leftrightsquigarrow \{(x, 0) \mid x \in \mathbb{R}\}\end{aligned}$$

3. Determine whether the set is a subspace of the vector space you listed in part 1. Justify your answer.

Problem 4. [Proving subspaces] For each of the following, prove that W is a subspace of V. For full points, your proofs should be neat, organized, and use full sentences.

(a) $V = \mathcal{P}_3(\mathbb{R}) = \{p(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$
$W = \{p(x) \in \mathcal{P}_3(\mathbb{R}) \mid p(0) = 0\}$.

(b) $V = \mathsf{M}_{2,2}(\mathbb{R}) = \left\{ \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$
$W = \left\{ \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathsf{M}_{2,2}(\mathbb{R}) \middle| a + d = 0 \right\}$.

(c) $V = \mathcal{C}[0,1] = \{\text{continuous functions } [0,1] \to \mathbb{R}\}$.
$W = \{f \in \mathcal{C}[0,1] \mid f(0) = f(1) = 0\}$.

Problem 5. [Computation-free linear combinations] Consider the following vectors in $\mathbb{R}^3$:

$$\tilde{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \tilde{\mathbf{y}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \tilde{\mathbf{z}} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \qquad \tilde{\mathbf{w}} = \begin{bmatrix} 6 \\ 2 \\ 8 \end{bmatrix}.$$

(a) Set up a system of linear equations to answer the following question: Is the vector $\tilde{\mathbf{b}} = (9, 5, 17) \in \mathbb{R}^3$ a linear combination of $\tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}}, \tilde{\mathbf{w}}$?

(b) Without performing any further computation, answer the following: Is $\tilde{\mathbf{b}} = (9, 5, 17)$ a linear combination of $\tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}}, \tilde{\mathbf{w}}$? Justify your answer.

(c) Fill in the blank: Given a list of vectors $\tilde{\mathbf{x_1}}, \ldots, \tilde{\mathbf{x_m}}$ in $\mathbb{R}^n$, every single vector $\tilde{\mathbf{b}} \in \mathbb{R}^n$ is a linear combination of $\tilde{\mathbf{x_1}}, \ldots, \tilde{\mathbf{x_m}}$ precisely when the RREF of the matrix with $\tilde{\mathbf{x_1}}, \ldots, \tilde{\mathbf{x_m}}$ as its columns has ____________________________.