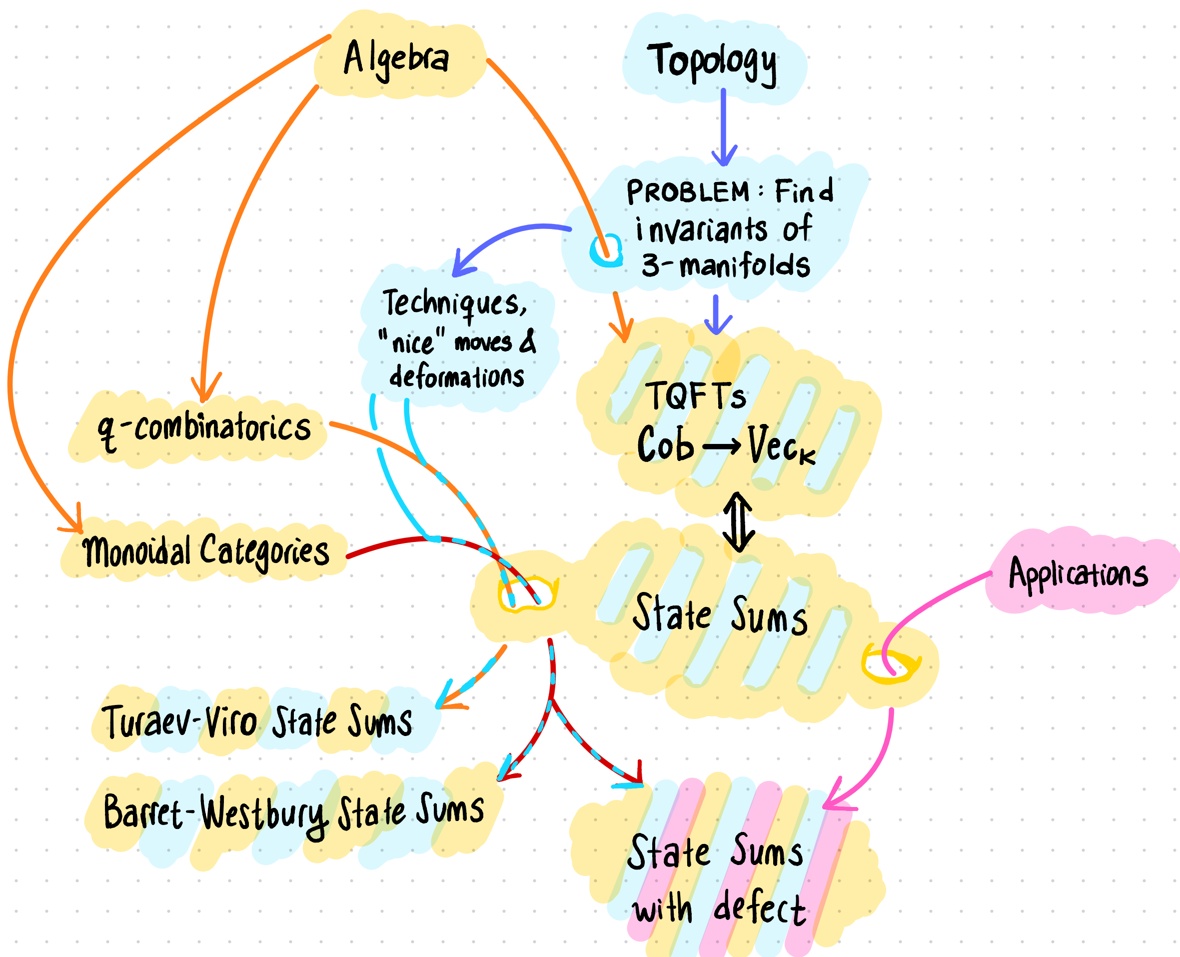
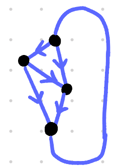


Algebra
Topology
PROBLEM: Find invariants of 3-manifolds
Techniques, "nice" moves & deformations
TQFTs
Cob ⟶ Vec$_K$
q-combinatorics
Monoidal Categories
State Sums
Applications
Turaev-Viro State Sums
Barret-Westbury State Sums
State Sums with defect

What are state sums?

STATE SUMS

	Turaev-Viro	Barrett-Westbury
1 Pick a set of labels/colors and weights for the labels	$I \subseteq \mathbb{R}$ $\omega_i \in \mathbb{C} : i \in I$	(simple) objects i in category $\dim(i)$
2 Define notion of "nice" triangle label ⇝ induces notion of "nice" tetrahedron label ⇝ induces notion of "nice" triangulation label	e.g. admissible triples (i,j,k) $i+j+k \in \mathbb{Z}$, triangle inequalities	e.g. use duals for the reverse orientations
3 Define a way to associate a scalar to nicely labeled tetrahedra (6j-symbol)	$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \in \mathbb{C}$	 $\in \mathrm{End}(\mathbb{1}) = \mathbb{C}$

⇝ induces a scalar associated to each nice triangulation labeling.

$$\boxed{\prod 6j\left(\text{labeled}\right) \cdot \prod_{\text{edges}} \text{weight}\left(\text{labeled edge}\right)}$$

call this a state

4 Require the 6j-symbols to satisfy axioms that make them independent under certain moves & triangulation changes.

Turaev-Viro

Certains choices with q-binomials & roots of unity work!

Barret-Westbury

Choices work if category is spherical, fusion, etc.

> **Remark** Basically all the work comes from finding examples that work & checking that they do work.

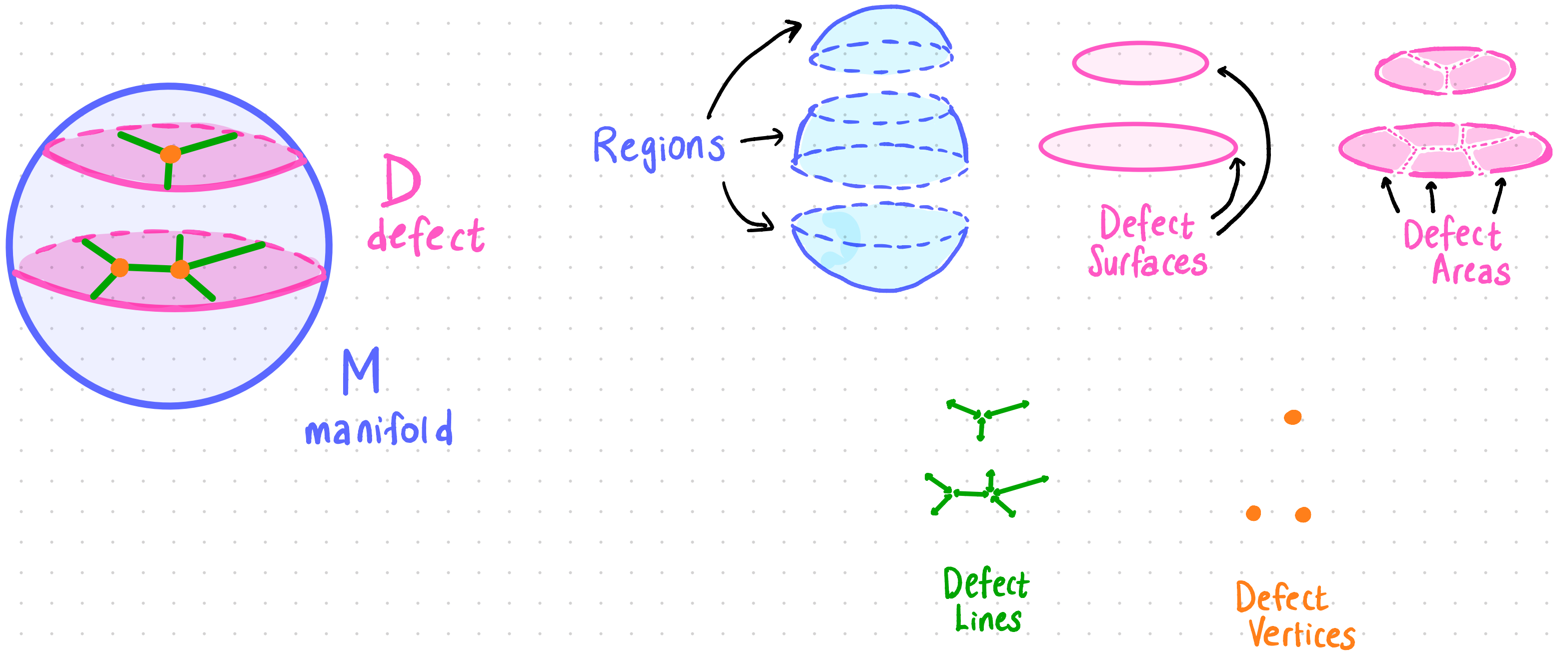
5 Obtain state sum by summing over all possible states (i.e. all possible labelings)

$$\boxed{Z(M) = \begin{pmatrix}\text{normalizing}\\ \text{weight}\end{pmatrix} \sum_{\text{labels}} \prod_{\triangle} 6j\begin{pmatrix}\text{labeled}\\ \triangle\end{pmatrix} \cdot \prod_{\text{edges}} \text{weight}\begin{pmatrix}\text{labeled}\\ \text{edge}\end{pmatrix}}$$

e.g. BW state sum is:

$$Z(M) = \frac{1}{(\dim \mathcal{C})^{\#\text{ vertices}}} \sum_{\substack{\ell \\ \text{labeling}}} \prod_{\triangle} 6j(\triangle) \cdot \prod_{\text{edges}} \dim\begin{pmatrix}\text{label of}\\ \text{edge}\end{pmatrix}$$

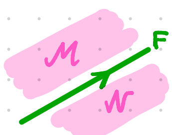
STATE SUM MODELS WITH DEFECT



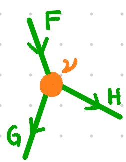
Big Idea: Extend labelings to include smaller dim. defect manifolds.
Label using higher "dimensional" categorical data.

Regions ⇝ $\mathcal{C}$ spherical fusion category

Oriented Defect Area ⇝ object of a 2-cat



Oriented Defect Line ⇝ 1-morphism of a 2-cat.



Defect Vertex ⇝ 2-morphism of a cat.

$\mathrm{Bimod}^{\theta}(\mathcal{C},\mathcal{D})$

module categories with trace

BIMODULE TRACES

Set-Up

$$\underset{\text{pivotal fusion}}{\mathcal{C}} \curvearrowright \underset{\text{finite semisimple}}{\mathcal{M}} \curvearrowleft \underset{\text{pivotal fusion}}{\mathcal{D}}$$

Idea: Give $\mathcal{M}$ a trace w/o pivotality.

$$\theta = \{\underbrace{\theta_m : \mathrm{End}_{\mathcal{M}}(m)}_{\text{each endomorph}} \longrightarrow \underbrace{\mathbb{C}}_{\text{gets a number}}\}_{m \in \mathcal{M}}$$

satisfying:

1. $\boxed{1}$ **cyclicity:** traces are symmetric.
2. $\boxed{2}$ **non-degeneracy:** For $m \in \mathcal{M}$ simple, want "$\dim(m)$" $:= \mathrm{trace}(1_m) \neq 0$.
3. $\boxed{3}$ **$\mathcal{C}$-compatible:** "$\mathrm{trace}(c \triangleright m) = \mathrm{trace}(m)\,\mathrm{trace}(c)$" so that $\dim(c \triangleright m) = \dim(c)\dim(m)$.

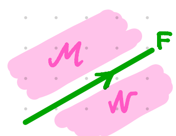
similarly for $\mathcal{D}$

THEOREM $\mathrm{Bimod}^{\theta}(\mathcal{C}, \mathcal{D})$ is a pivotal 2-cat.

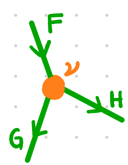
MEUSBERGER STATE SUM

Regions $\rightsquigarrow$ $\mathcal{C}$ spherical fusion category

Oriented Defect Area $\underset{\text{a 2-cat}}{\overset{\text{object of}}{\rightsquigarrow}}$ $\mathcal{M} \in \mathrm{Bimod}^{\theta}(\mathcal{C},\mathcal{D})$



Oriented Defect Line $\underset{\text{of a 2-cat.}}{\overset{\text{1-morphism}}{\rightsquigarrow}}$ $\mathcal{M} \xrightarrow{F} \mathcal{N}$ a $(\mathcal{C},\mathcal{D})$-bimod. functor

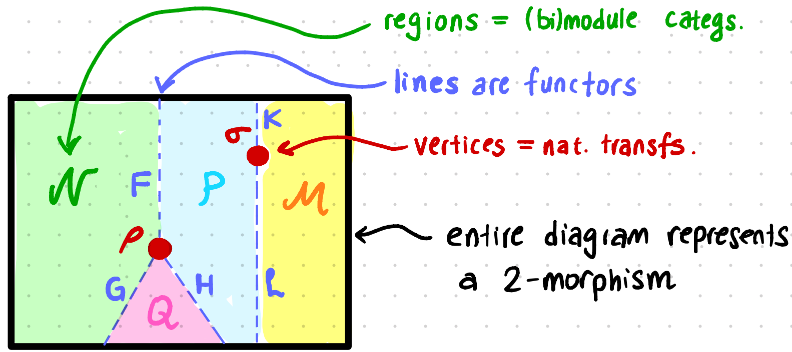


Defect Vertex $\underset{\text{of a cat.}}{\overset{\text{2-morphism}}{\rightsquigarrow}}$ cyclic equiv class of $(\mathcal{C},\mathcal{D})$-bimodule nat. transf. (e.g. $F \overset{\nu}{\Rrightarrow} GH$).

Aside: Diagrams
Diagrams
Diagrams
Diagrams
Diagrams

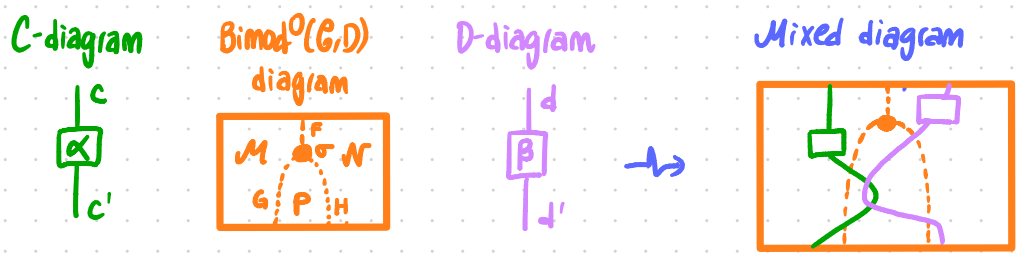
DIAGRAMS

2-Category diagrams for $\mathrm{Bimod}^{\Theta}(\mathcal{C},\mathcal{D})$



Mixed Diagrams for $\mathrm{Bimod}^{\Theta}(\mathcal{C},\mathcal{D})$

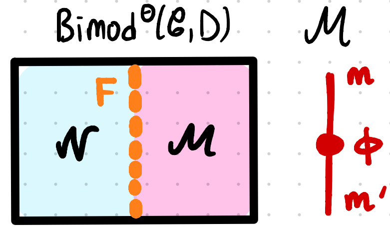
Layer $\mathcal{C}$, $\mathcal{D}$, and $\mathrm{Bimod}^{\Theta}(\mathcal{C},\mathcal{D})$ diagrams



Reinterpret as $\mathrm{Bimod}^{\Theta}(\mathcal{C},\mathcal{D})$ diagram using:

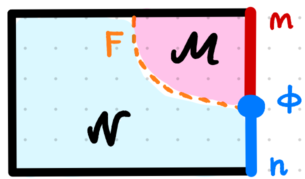
$$c \xrightarrow{f} c' \quad \rightsquigarrow \quad \boxed{M \xrightarrow{c\triangleright -} M} \overset{f\triangleright -}{\Downarrow} \boxed{M \xrightarrow{c'\triangleright -} M}$$

Bordered Diagrams for $\mathrm{Bimod}^{\ominus}(\mathcal{C},\mathcal{D})$



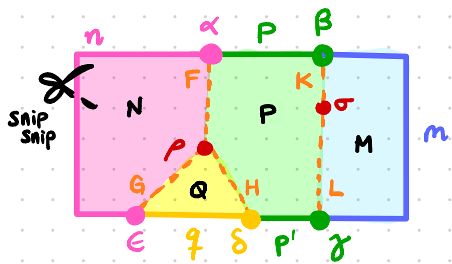
represents

$$\begin{tikzcd} Fm \arrow[d, "F\phi"'] \\ Fm' \end{tikzcd}$$



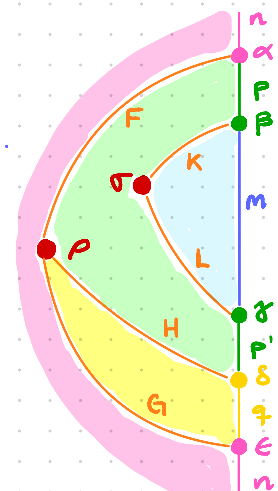
represents

$$\begin{tikzcd} Fm \arrow[d, "F\phi"'] \\ n \end{tikzcd}$$



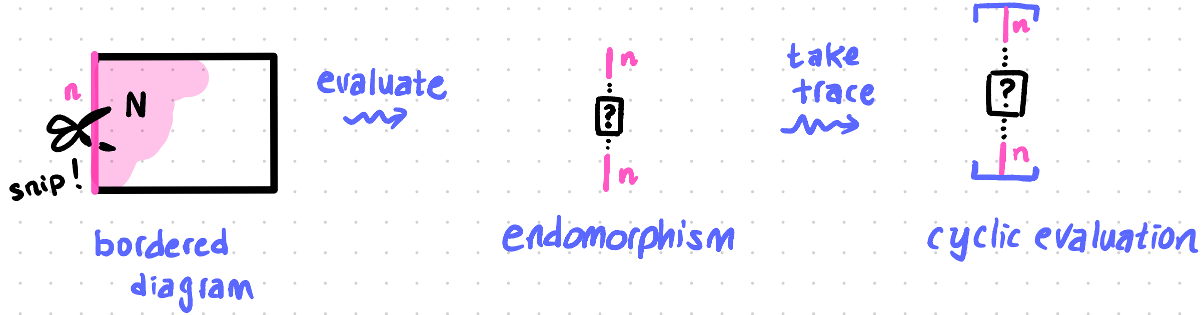
represents...

* various ways of "spreading out" all equal by naturality



$$\begin{aligned}
&\left.\begin{array}{l} n \xrightarrow{\alpha} Fp \xrightarrow{F\beta} FKm \end{array}\right\} \text{top} \\
&\left.\begin{array}{l} FKm \xrightarrow{F\sigma_m} FLm \xrightarrow{\rho_{Lm}} GHLm \end{array}\right\} \nu = \rho L \circ F\sigma \\
&\left.\begin{array}{l} GHLm \xrightarrow{GH\gamma} GHp' \xrightarrow{G\delta} Gq \xrightarrow{\epsilon} n \end{array}\right\} \text{bottom}
\end{aligned}$$

Polygon Diagrams for $\mathrm{Bimod}^{\Theta}(\mathcal{C},\mathcal{D})$

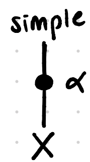


- Better invariants (e.g. using pivotality to bend strings)
- Doesn't matter where we cut.
- Just a number!

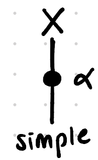
PROJECTIONS & INCLUSIONS

Idea Pick nice bases for Hom (X , simple) , Hom (simple , X)
to show how polygon diagrams glue.

notation use $\alpha, \beta, \gamma, \cdots$ to index the bases



= α-th inclusion simple $\to X$

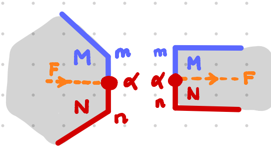


= α-th projection simple $\to X$

Gluing Identities

(Gluing sides)

$\sum_\alpha$



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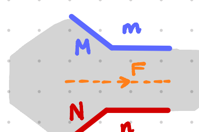
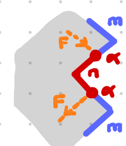


(Glue around a vertex)

$\sum_{n,\alpha} \dim(n)$



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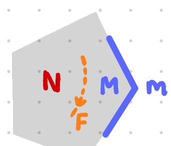
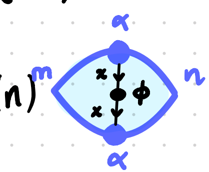
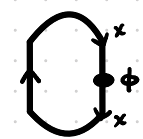


(Glue a 2-gon)

$\sum_{m,n,\alpha} \dim(m)\dim(n)$

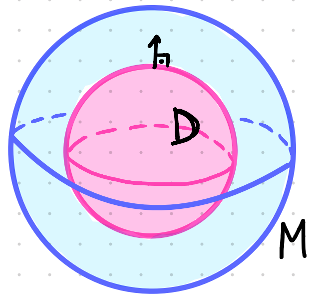


= $\dim(\mathcal{M})$



Back to state sums

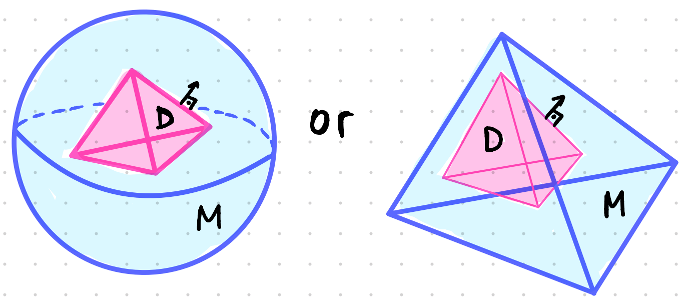
EXAMPLE UNKNOTTED 3-BALL W/ DEFECT SPHERE



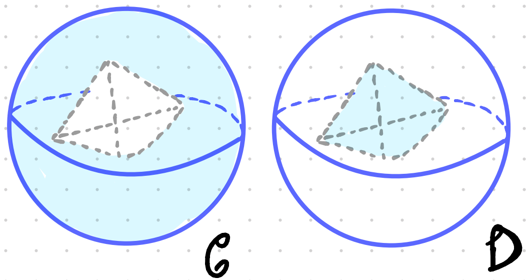
A solid ball w/ a hollow sphere inside as defect.

No defect lines or vertices.

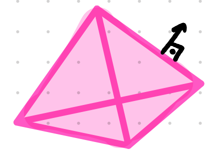
Assume the sphere is unknotted, i.e. we can PL Homeo to make the sphere a tetrahedron.



EXAMPLE UNKNOTTED 3-BALL W/ DEFECT SPHERE



Two regions



$M \in \mathrm{Bimod}^{\theta}(\mathcal{C}, \mathcal{D})$

Defect surface

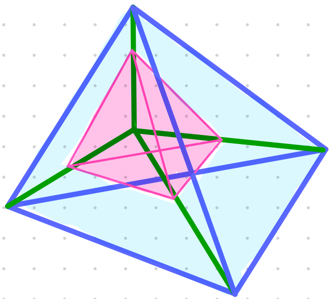
one defect area

no defect lines

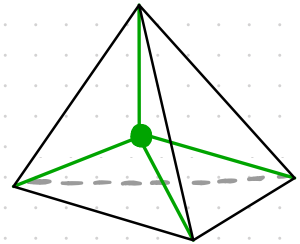
no defect vertex

EXAMPLE UNKNOTTED 3-BALL W/ DEFECT SPHERE

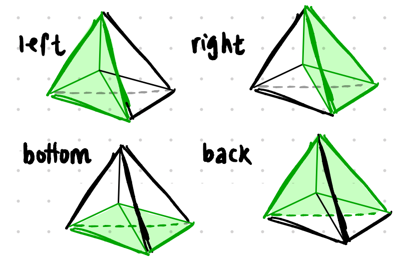
Pick triangulation



$(M, \partial M, T, D)$

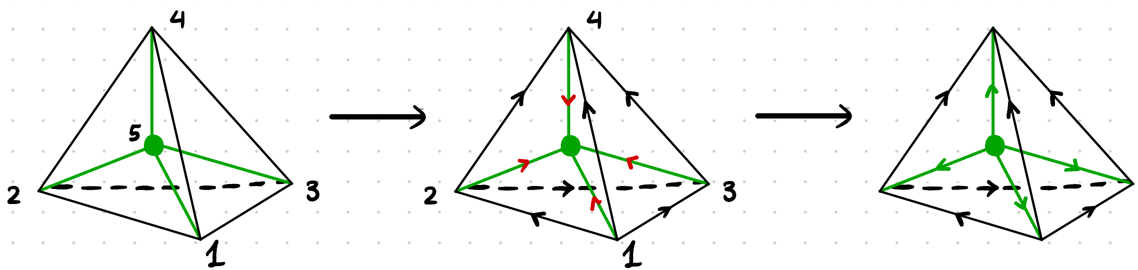


$(M, \partial M, T)$



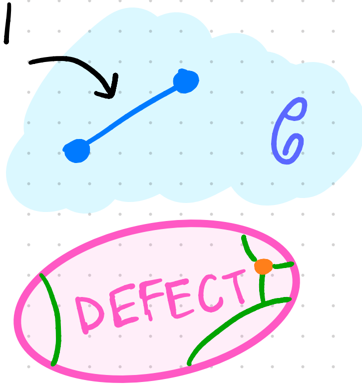
tetrahedra of triangulation

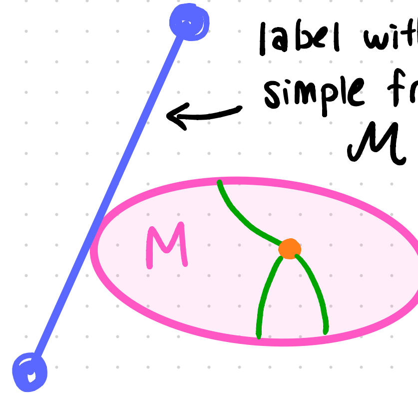
Orient



LABELING TRIANGULATIONS

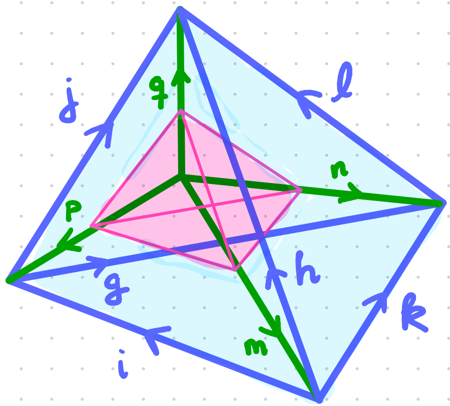
label w/ simple from $\mathcal{C}$

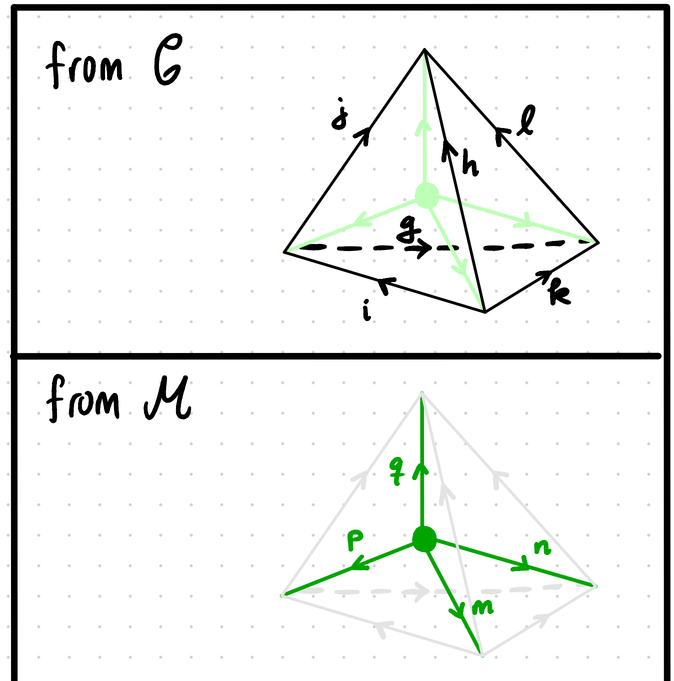




label with simple from $\mathcal{M}$

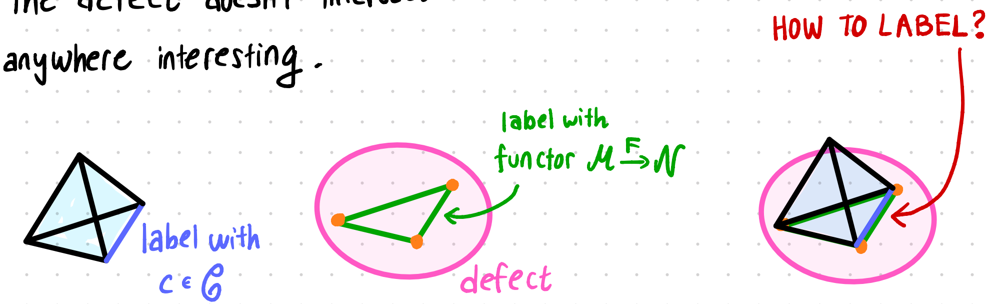
EXAMPLE UNKNOTTED 3-BALL W/ DEFECT SPHERE





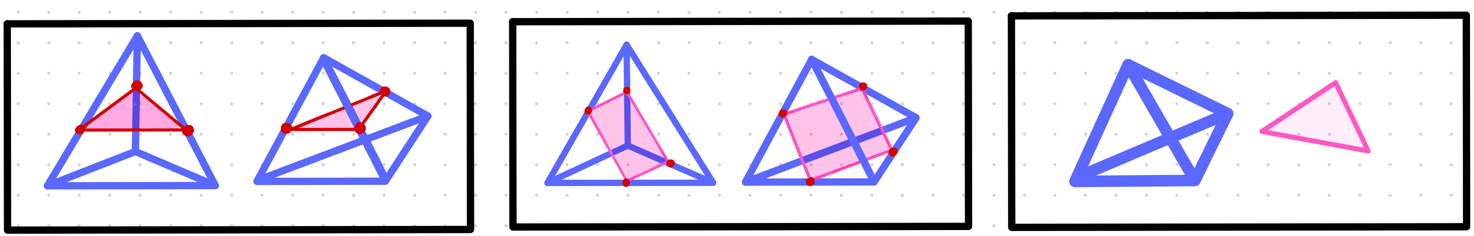
Big Idea: Adding the defect manifolds labeling shouldn't be too bad if the defect doesn't intersect the triangulation anywhere interesting.

e.g. don't want



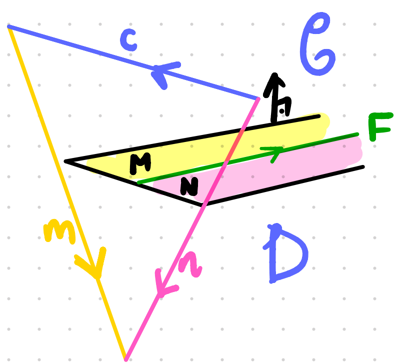
$\longrightarrow$ use generic transversal triangulations.

Proposition Any triangulation can be perturbed to a generic + transversal one.



ASSIGNING HOM-SPACES TO TRIANGLES

Modulo orientations, all triangles intersecting defect look like:



$$c,\ m,\ n,\ M \xrightarrow{F} N$$

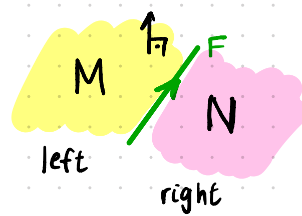
Combine the two edges w/ same direction

$$c \triangleright m,\ n,\ M \xrightarrow{F} N$$

Apply functor so objs are in same place

$$F(c \triangleright m),\ n$$

"The left one" wrt to defect normal is domain

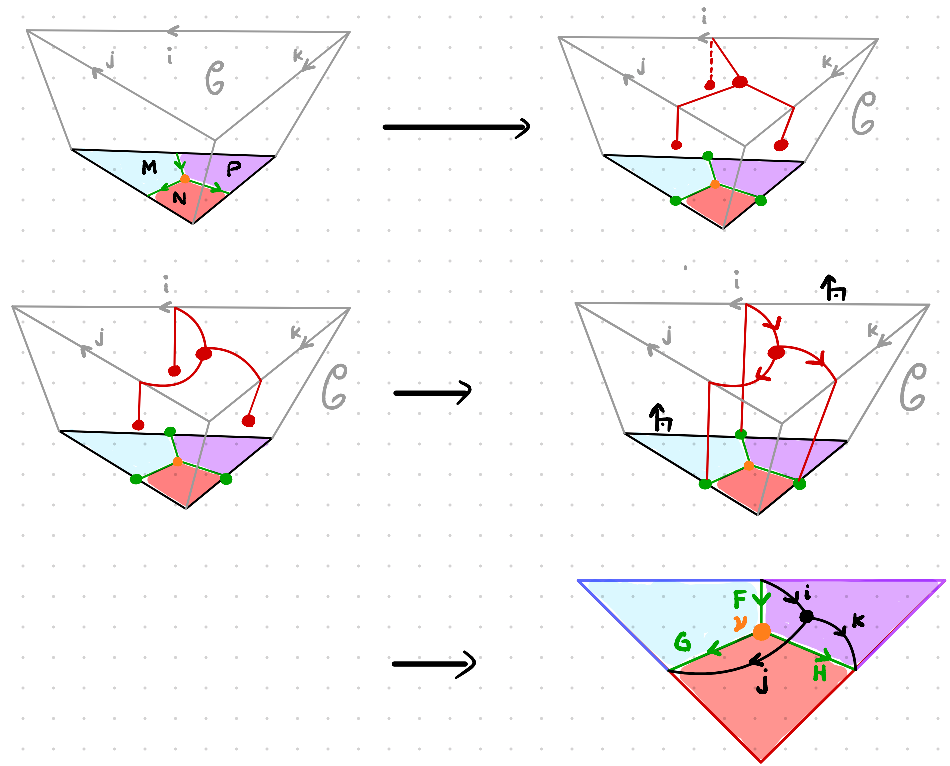


$$\mathrm{Hom}(F(c \triangleright m), n)$$

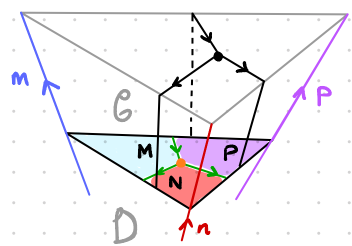
6j - Symbols

Draw dual graph for edges that don't touch defect

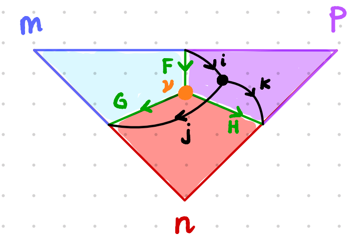
Curve, orient nicely & project to defect

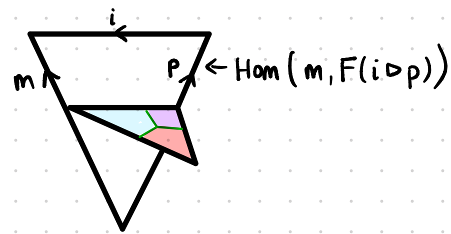


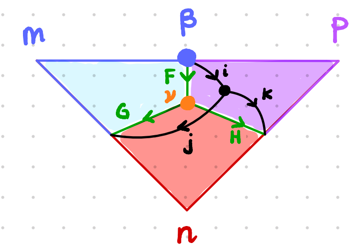
Label borders:



$\rightsquigarrow$

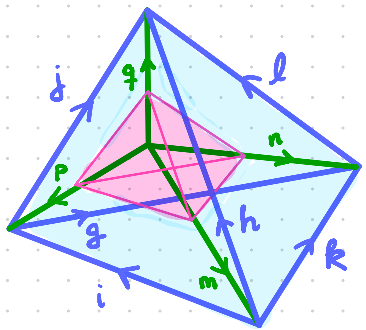


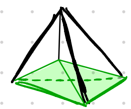


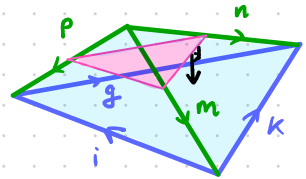


$$6j(\text{tetrahedron}) = ev(P)$$

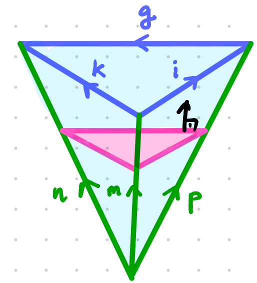
EXAMPLE UNKNOTTED 3-BALL W/ DEFECT SPHERE



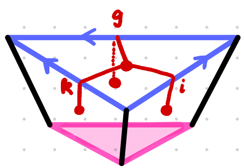




rotate



draw dual edges for edges not touching defect

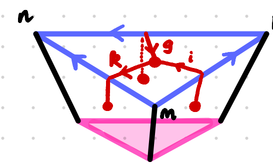


curve

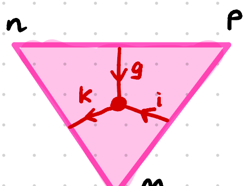
(not needed)

X

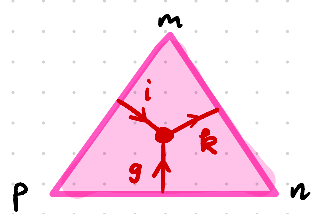
orient



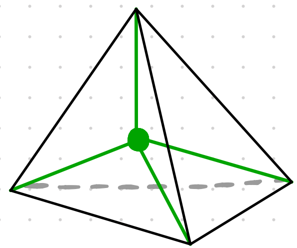
project to defect

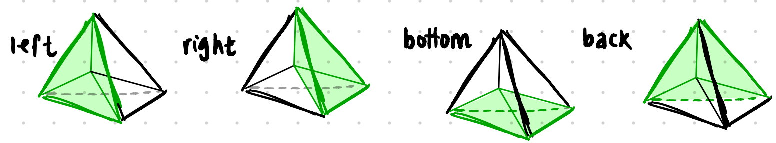


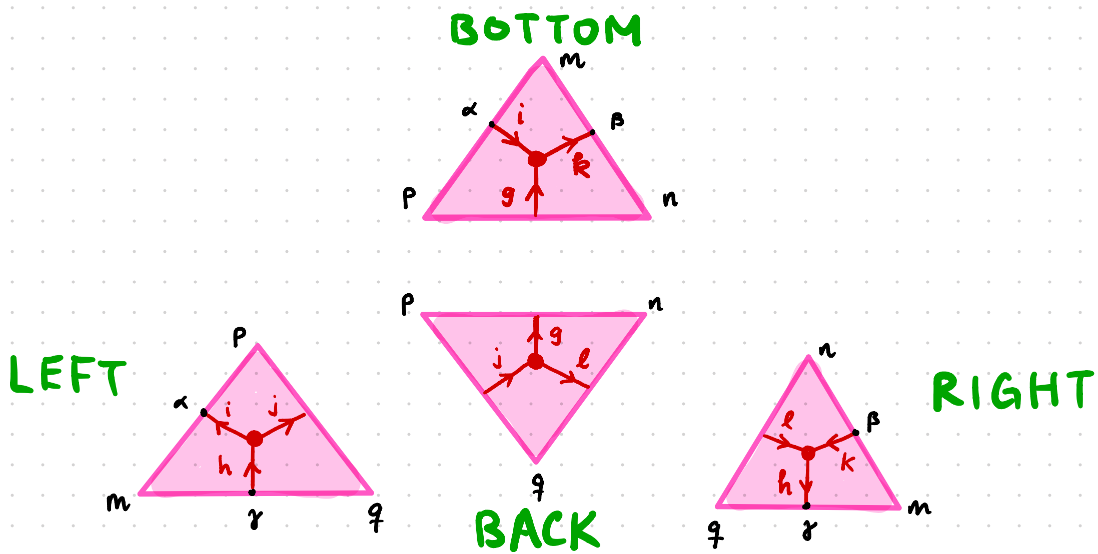
rotate



EXAMPLE UNKNOTTED 3-BALL W/ DEFECT SPHERE







STATE SUM W/ DEFECT

$$Z'(M, \ell_{\partial M}, b) := \frac{1}{\prod_{r \in R} \prod_{v \in V_r} \dim(\mathcal{C}_r)} \sum_{\ell_i : \ell \in \mathcal{L}} \sum_{\mathrm{Hom}(\Delta_i)} \prod_{t \in \mathrm{Tet}} 6j(t) \prod_{e \in E} \dim(\ell(e))$$

weight

sum over all internal labelings

sum over all choices of projections + inclusions for internal triangles

6j

weights

To make it work with gluing, change $\dim(\mathcal{C}_r)$, $\dim(\ell(e)) \to \sqrt{\dim(\mathcal{C}_r)}, \sqrt{\dim(\ell(e))}$ for boundary points + edges.

$$Z(M, \ell_{\partial M}, b) = \frac{1}{\prod_{r \in R} \prod_{v \in V_r} \dim(\mathcal{C}_r)^{\epsilon(v)}} \sum_{\ell_i : \ell \in \mathcal{L}} \sum_{\mathrm{Hom}(\Delta_i)} \prod_{t \in \mathrm{Tet}} 6j(t) \prod_{e \in E} \dim(\ell(e))^{\epsilon(e)}$$

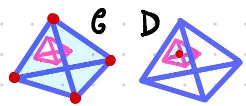
$= \frac{1}{2}$ for ∂ edges

EXAMPLE UNKNOTTED 3-BALL W/ DEFECT SPHERE

$$Z'(M, l_{\partial M}, b) := \frac{1}{\prod_{r \in R} \prod_{v \in V_r} \dim(\mathcal{C}_r)} \sum_{\substack{\text{labels}\\ \text{on}\\ \text{internal}\\ \text{edges}}} \sum_{\substack{\text{assignments}\\ \text{of } p/j\text{'s}\\ \text{to internal}\\ \text{triangles}}} \prod_{\substack{\text{Tetrahedra}\\ t}} 6j(t) \prod_{\substack{\text{edges}\\ e}} \dim(l(e))$$

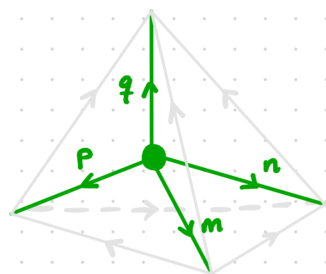
Break it down:

$$\frac{1}{\prod_{r \in R} \prod_{v \in V_r} \dim(\mathcal{C}_r)} \xrightarrow{\substack{\text{Two regions: } \mathcal{C}, \mathcal{D}\\ 4 \text{ vertices in } \mathcal{C}\\ 1 \text{ vertex in } \mathcal{D}}} \frac{1}{\dim(\mathcal{C})^4 \dim(\mathcal{D})}$$



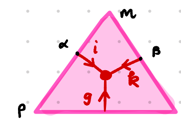
$$Z'(M, l_{\partial M}, b) := \frac{1}{\prod_{r \in R} \prod_{v \in V_r} \dim(\mathcal{C}_r)} \sum_{\substack{\text{labels on}\\ \text{internal}\\ \text{edges}}} \sum_{\substack{\text{assignments}\\ \text{of } p/j\text{'s}\\ \text{to internal}\\ \text{triangles}}} \prod_{\substack{\text{Tetrahedra}\\ t}} 6j(t) \prod_{\substack{\text{edges}\\ e}} \dim(l(e))$$

$$\sum_{\substack{\text{labels on}\\ \text{internal edges}}} \xrightarrow[]{\substack{\text{four internal edges}\\ m,n,p,q}} \sum_{\substack{m,n,\\ p,q}}$$

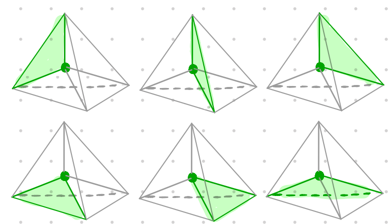


$$Z'(M, \ell_{\partial M}, b) := \frac{1}{\prod\limits_{r\in R}\prod\limits_{v\in V_r}\dim(C_r)} \sum_{\substack{\text{labels}\\ \text{on}\\ \text{internal}\\ \text{edges}}} \sum_{\substack{\text{assignments}\\ \text{of } p/j\text{'s}\\ \text{to internal}\\ \text{triangles}}} \prod_{\substack{\text{Tetrahedra}\\ t}} 6j(t) \prod_{\substack{\text{edges}\\ e}} \dim(\ell(e))$$

$6j(t)$ ← tetrahedron $\xrightarrow{\text{def}}$ $= ev(P) =$

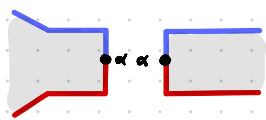


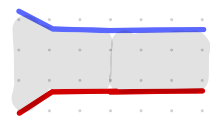
$\sum\limits_{\substack{\text{assignments}\\ \text{of } p/j\text{'s}\\ \text{to internal}\\ \text{triangles}}}$ $\xrightarrow{\text{six internal triangles}}$ $\sum\limits_{\substack{\alpha,\beta,\gamma,\\ \delta,\sigma,\tau}}$



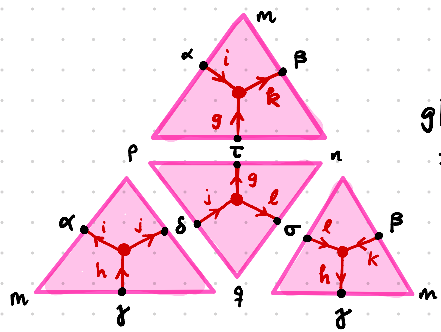
$$Z'(M,\ell_{\partial M},b) := \frac{1}{\prod_{r\in R}\prod_{v\in V_r}\dim(\mathcal{C}_r)} \sum_{\substack{\text{labels}\\ \text{on}\\ \text{internal}\\ \text{edges}}} \sum_{\substack{\text{assignments}\\ \text{of p/j's}\\ \text{to internal}\\ \text{triangles}}} \prod_{\substack{\text{Tetrahedra}\\ t}} 6j(t) \prod_{\substack{\text{edges}\\ e}} \dim(\ell(e))$$

Recall "gluing sides" for polygon diagrams:

$\sum_{\alpha}$ 

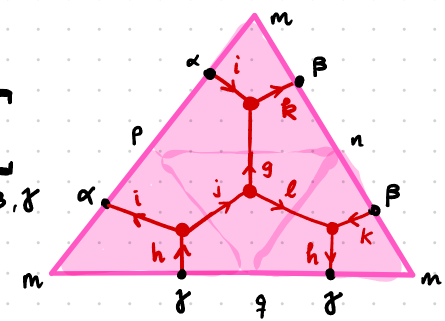
$=$ 

$$\sum_{\substack{\text{assignments}\\ \text{of p/j's}\\ \text{to internal}\\ \text{triangles}}} \prod_{\substack{\text{Tetrahedra}\\ t}} 6j(t) = \sum_{\substack{\alpha,\beta,\gamma,\\ \delta,\sigma,\tau}}$$



glue
$=$

$$\sum_{\alpha,\beta,\gamma}$$



$$Z'(M, \ell_{\partial M}, b) := \frac{1}{\prod\limits_{r\in R} \prod\limits_{v\in V_r} \dim(\mathcal{C}_r)} \sum_{\substack{\text{labels}\\ \text{on}\\ \text{internal}\\ \text{edges}}} \sum_{\substack{\text{assignments}\\ \text{of p/j's}\\ \text{to internal}\\ \text{triangles}}} \prod_{\substack{\text{Tetrahedra}\\ t}} 6j(t) \prod_{\substack{\text{edges}\\ e}} \dim(\ell(e))$$

$$\prod_{\substack{\text{edges}\\ e}} \dim(\ell(e)) = \underbrace{\prod_{\substack{\text{edges}\\ \text{labelled w/}\\ \mathcal{C}}} \dim(\ell(e))}_{\text{pull out and use}} \underbrace{\prod_{\substack{\text{edges}\\ \text{labelled w/}\\ \mathcal{M}}} \dim(\ell(e))}_{\text{inner edges } m,n,p,q}$$

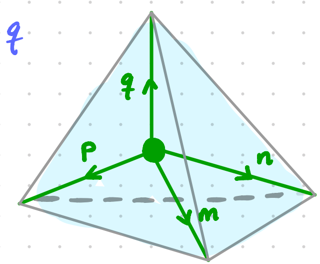
pull out and use

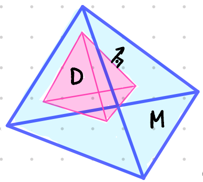
$$\dim(\mathcal{C}) = \sum_{\substack{x \\ \text{simples in } \mathcal{C}}} \dim(x)^2$$

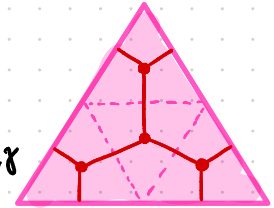
to cancel out the $\frac{1}{\dim(\mathcal{C})}$'s

inner edges m, n, p, q

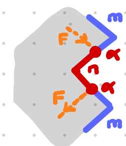
$$= \dim(m)\dim(n)\dim(p)\dim(q)$$

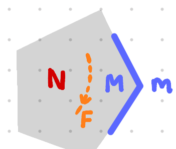


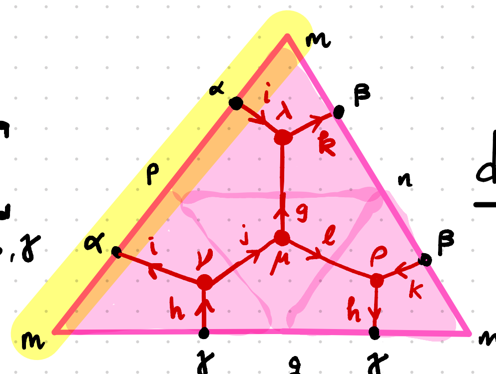
$$Z'\Bigg($$ 

$$\Bigg) = \frac{1}{\dim(D)} \sum_{m,n,p,q} \sum_{\alpha,\beta,\gamma}$$  $$\dim(m)\dim(n)\dim(p)\dim(q).$$

Recall "glue around a vertex" for polygon diagrams:

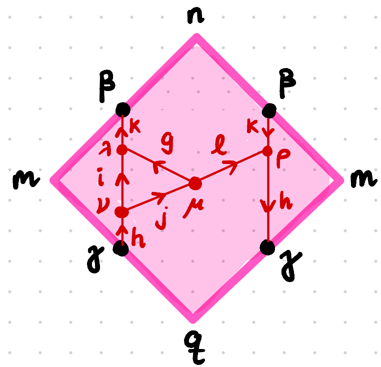
$$\sum_{n,\alpha} \dim(n)$$ 

$$=$$ 

$$= \sum_{\substack{m,n\\p,q}} \sum_{\alpha,\beta,\gamma}$$ 

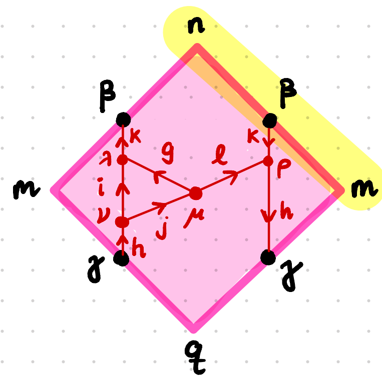
$$\frac{\dim(m)\dim(n)\dim(p)\dim(q)}{\dim(D)}$$

$$= \sum_{m,n,q} \sum_{\beta,\gamma}$$



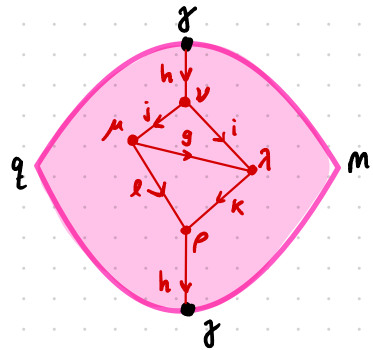
$$\frac{\dim(m)\dim(n)\dim(q)}{\dim(D)}$$

$$= \sum_{m,n,q} \sum_{\beta,\gamma}$$



$$\frac{\dim(m)\dim(n)\dim(q)}{\dim(D)}$$

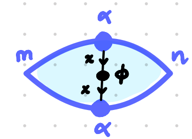
$= \sum_{m,q,\gamma}$



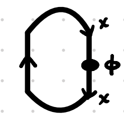
$\dfrac{\dim(m)\,\dim(q)}{\dim(D)}$

Recall "gluing a 2-gon":

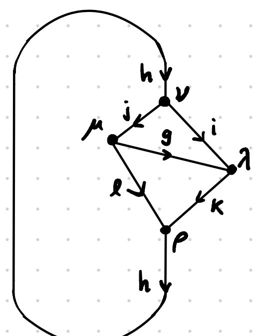
$\sum_{m,n,\alpha} \dim(m)\dim(n)$



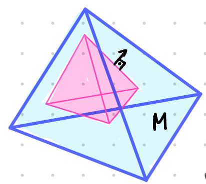
$= \dim(\mathcal{M})$

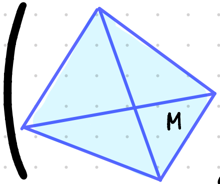


$= \dfrac{\dim(\mathcal{M})}{\dim(D)}$



← Barret-Westbury
6j for 

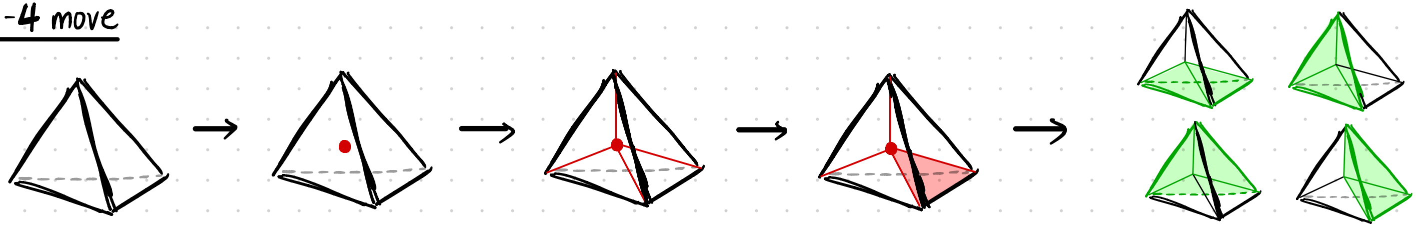
$Z'\Big($ 

$\Big) = \dfrac{\dim(\mathcal{M})}{\dim(D)}\, Z'\Big($ 

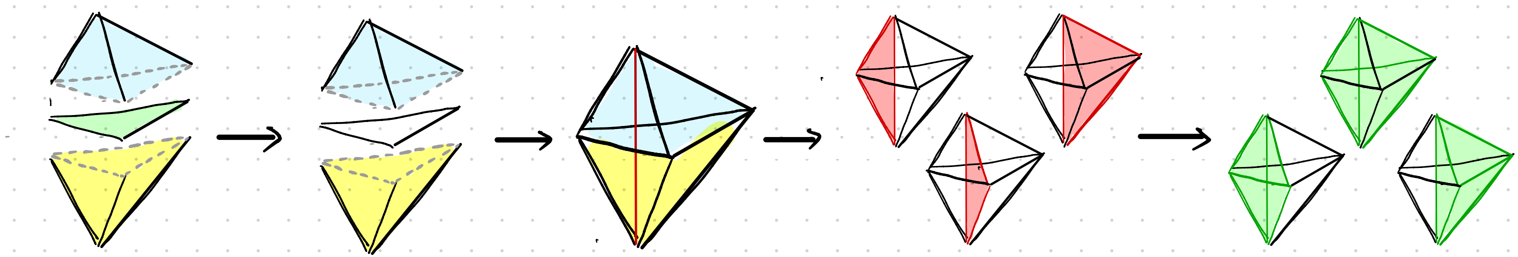
$\Big)$

TRIANGULATION INDEPENDENCE

1-4 move

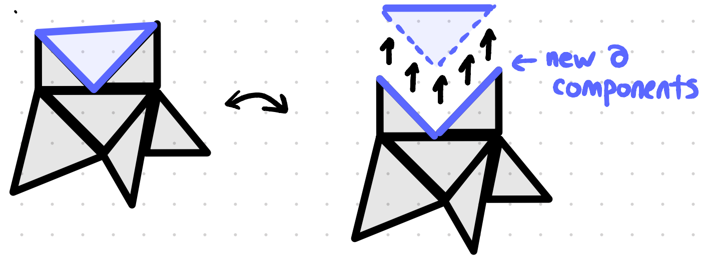


2-3 move

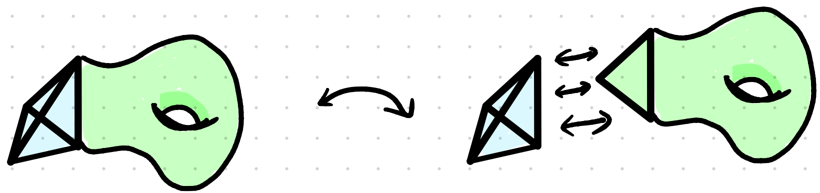


Elementary Shellings

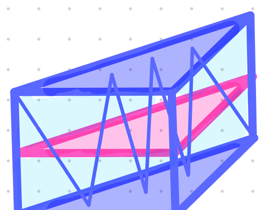
n=2



n=3



FINE NEIGHBORHOODS

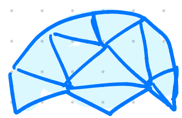


generic
transversal

$\cong [0,1] \times D$

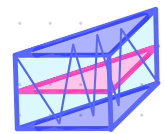
TRIANGULATION INDEPENDENCE

Idea: Use bistellar moves to refine triangulations & split into chunks that are all



no defect

or



fine ngbd w/ defect disk

Show the latter have triangulation + bistellar invariant state sums.

Glue everything back to diff. triangulations while checking state sum remains the same.