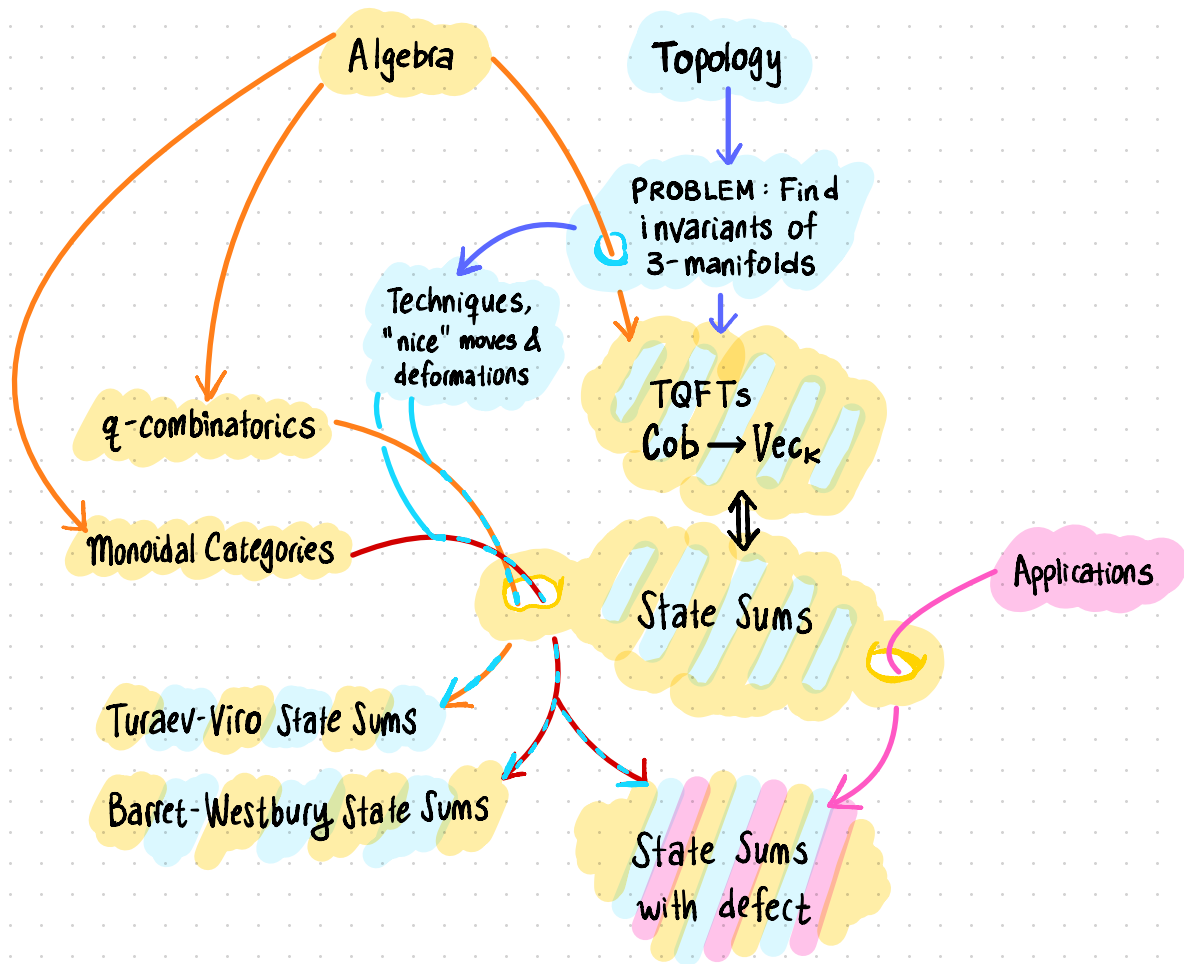




What are state sums?

STATE SUMS

- 1 Pick a set of labels/colors
and weights for the labels
- 2 Define notion of "nice" triangle label
 $\leadsto$ induces notion of "nice" tetrahedron label
 $\leadsto$ induces notion of "nice" triangulation label

- 3 Define a way to associate a scalar
to nicely labeled tetrahedra ($6j$ -symbol)

$\leadsto$ induces a scalar associated to each
nice triangulation labeling.

Turaev-Viro

$$I \subseteq \mathbb{R}$$

$$\omega_i \in \mathbb{C} : i \in I$$

{ e.g. admissible
triples (i, j, k)
 $i + j + k \in \mathbb{Z}$,
triangle inequalities

$$\begin{vmatrix} i & j & k \\ \ell & m & n \end{vmatrix} \in \mathbb{C}$$

Barret-Westbury

(simple) objects i in category
 $\dim(i)$

{ e.g. use duals for the
reverse orientations

$$\left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right) \in \text{End}(\mathbb{1}) = \mathbb{C}$$

$$\prod_{\text{tetrahedra}} 6j \left(\begin{array}{c} \text{labeled} \\ \text{tetrahedron} \end{array} \right) \cdot \prod_{\text{edges}} \text{weight} \left(\begin{array}{c} \text{labeled} \\ \text{edge} \end{array} \right)$$

$\uparrow$ call this a **State**

- 4 Require the $6j$ -symbols to satisfy axioms that make them independent under certain moves & triangulation changes.

Turaev-Viro

Certain choices with q -binomials & roots of unity work!

Barret-Westbury

Choices work if category is spherical, fusion, etc.

Remark Basically all the work comes from finding examples that work & checking that they do work.

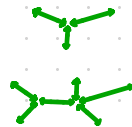
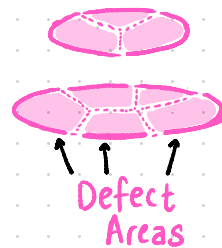
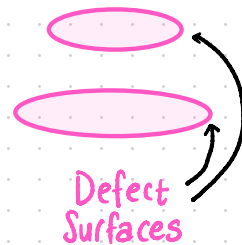
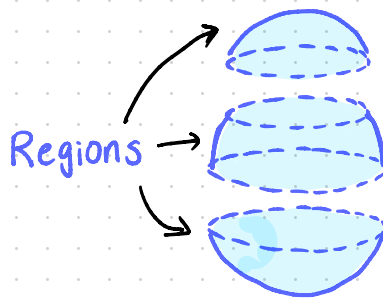
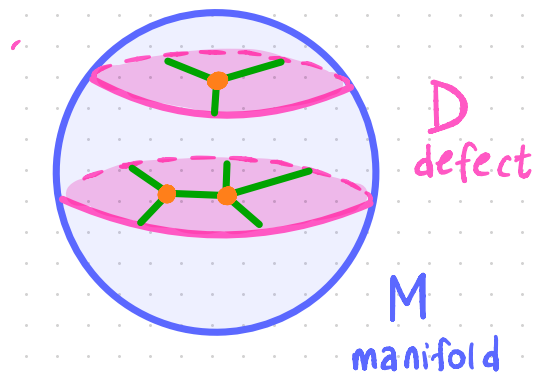
- 5 Obtain state sum by summing over all possible states (i.e. all possible labelings)

$$Z(M) = \left(\frac{\text{normalizing}}{\text{weight}} \right) \sum_{\text{labels}} \prod_{\triangle} 6j(\triangle^{\text{labeled}}) \cdot \prod_{\text{edges}} \text{weight}(\triangle^{\text{labeled}})$$

e.g. BW state sum is:

$$Z(M) = \frac{1}{(\dim \mathcal{C})^{\# \text{vertices}}} \sum_{\substack{\ell \\ \text{labeling}}} \prod_{\triangle} 6j(\triangle) \cdot \prod_{\text{edges}} \dim(\text{label of edge})$$

STATE SUM MODELS WITH DEFECT



Defect
Lines



Defect
Vertices

Big Idea : Extend labelings to include smaller dim. defect manifolds.
 Label using higher "dimensional" categorical data.



Regions



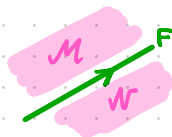
$\mathcal{G}$ spherical fusion category



Oriented
Defect
Area



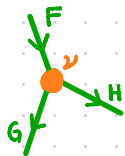
object of a 2-cat



Oriented
Defect
Line



1-morphism of a 2-cat.



Defect
Vertex

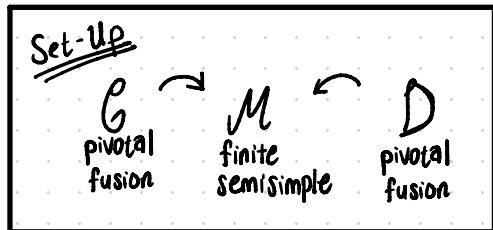


2-morphism of a cat.

$\text{Bimod}^0(\mathcal{G}, \mathcal{D})$

module categories
with trace

BIMODULE TRACES



Idea: Give $\mathcal{M}$ a trace w/o pivotality.

$$\theta = \{ \theta_m : \underset{\substack{\uparrow \\ \text{each endomorph}}}{\text{End}_{\mathcal{M}}(m)} \longrightarrow \underset{\substack{\uparrow \\ \text{gets a number}}}{\mathbb{C}} \}_{m \in \mathcal{M}}$$

satisfying:

1 cyclicity: traces are symmetric.

2 non-degeneracy: For $m \in \mathcal{M}$ simple, want " $\dim(m)$ " := $\text{trace}(1_m) \neq 0$.

3 $\mathcal{C}$ -compatible: " $\text{trace}(c \triangleright m) = \text{trace}(m) \text{trace}(c)$ "
so that $\dim(c \triangleright m) = \dim(c) \dim(m)$.
} similarly for $\mathcal{D}$

THEOREM $\text{Bimod}^{\theta}(\mathcal{C}, \mathcal{D})$ is a pivotal 2-cat.

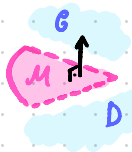
MEUSBERGER STATE SUM



Regions



$\mathcal{G}$ spherical fusion category



Oriented
Defect
Area

object of
a 2-cat

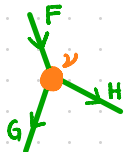
$\mathcal{M} \in \text{Bimod}^{\theta}(\mathcal{G}, \mathcal{D})$



Oriented
Defect
Line

1-morphism
of a 2-cat.

$\mathcal{M} \xrightarrow{F} \mathcal{N}$ a $(\mathcal{G}, \mathcal{D})$ -bimod. functor



Defect
Vertex

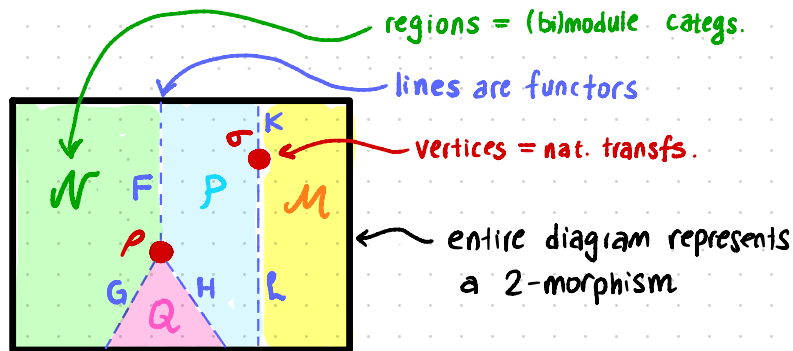
2-morphism
of a cat.

cyclic equiv class of $(\mathcal{G}, \mathcal{D})$ -bimodule
nat. transf. (e.g. $F \xrightarrow{\gamma} GH$).

Aside: Diagrams
Diagrams
Diagrams
Diagrams
Diagrams

DIAGRAMS

2-category diagrams
for $\text{Bimod}^0(\mathcal{C}, \mathcal{D})$



Mixed Diagrams
for $\text{Bimod}^0(\mathcal{C}, \mathcal{D})$

Layer $\mathcal{C}$, $\mathcal{D}$, and $\text{Bimod}^0(\mathcal{C}, \mathcal{D})$ diagrams

$\mathcal{C}$ -diagram



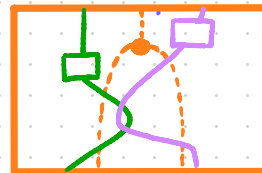
$\text{Bimod}^0(\mathcal{C}, \mathcal{D})$
diagram



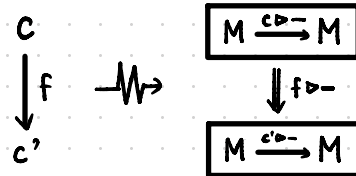
$\mathcal{D}$ -diagram



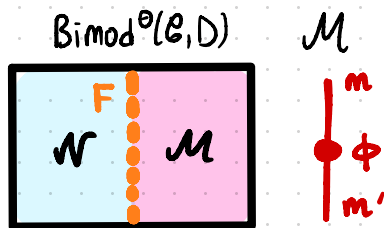
Mixed diagram



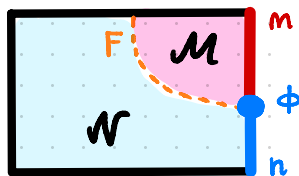
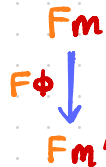
Reinterpret as $\text{Bimod}^0(\mathcal{C}, \mathcal{D})$ diagram using:



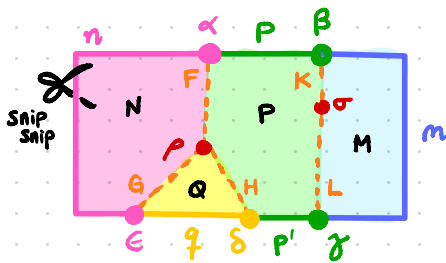
Bordered Diagrams for $\text{Bimod}^0(\mathcal{C}, \mathcal{D})$



represents



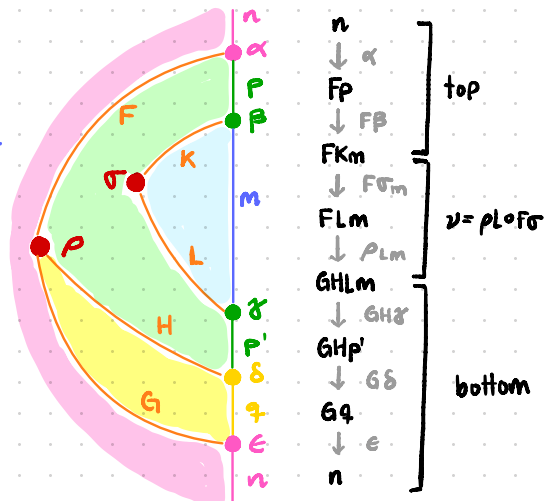
represents



represents...



*various ways of "spreading out" all equal by naturality



Polygon Diagrams for $\text{Bimod}^{\Theta}(\mathcal{C}, \mathcal{D})$



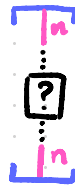
bordered
diagram

evaluate
→



endomorphism

take
trace
→



cyclic evaluation

- Better invariants (e.g. using pivotality to bend strings)
- Doesn't matter where we cut.
- Just a number!

PROJECTIONS & INCLUSIONS

Idea Pick nice bases for $\text{Hom}(X, \text{simple})$, $\text{Hom}(\text{simple}, X)$ to show how polygon diagrams glue.

notation use $\alpha, \beta, \gamma, \dots$ to index the bases

$$\begin{array}{c} \text{simple} \\ | \\ \bullet \alpha \\ | \\ X \end{array} = \begin{array}{c} \alpha\text{-th inclusion} \\ \text{simple} \rightarrow X \end{array} \qquad \begin{array}{c} X \\ | \\ \bullet \alpha \\ | \\ \text{simple} \end{array} = \begin{array}{c} \alpha\text{-th projection} \\ \text{simple} \rightarrow X \end{array}$$

Gluing Identities

(Gluing sides)

$$\sum \alpha \quad \begin{array}{c} \text{Diagram 1: A polygon with a red side labeled } n \text{ and a blue side labeled } m. \text{ A dashed orange line labeled } F \text{ connects two vertices. A red line labeled } N \text{ and a blue line labeled } M \text{ are also shown.} \end{array} = \begin{array}{c} \text{Diagram 2: A simplified polygon with a red side labeled } n \text{ and a blue side labeled } m. \text{ A dashed orange line labeled } F \text{ connects two vertices. A red line labeled } N \text{ and a blue line labeled } M \text{ are also shown.} \end{array}$$

(Glue around a vertex)

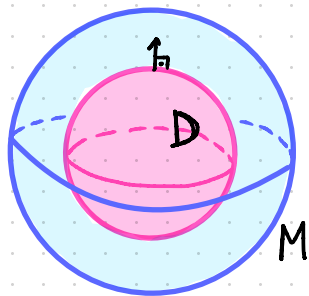
$$\sum_{n, \alpha} \dim(n) \quad \begin{array}{c} \text{Diagram 1: A polygon with a red side labeled } n \text{ and a blue side labeled } m. \text{ A dashed orange line labeled } F \text{ connects two vertices. A red line labeled } N \text{ and a blue line labeled } M \text{ are also shown.} \end{array} = \begin{array}{c} \text{Diagram 2: A simplified polygon with a red side labeled } n \text{ and a blue side labeled } m. \text{ A dashed orange line labeled } F \text{ connects two vertices. A red line labeled } N \text{ and a blue line labeled } M \text{ are also shown.} \end{array}$$

(Glue a 2-gon)

$$\sum_{m, n, \alpha} \dim(m) \dim(n) \quad \begin{array}{c} \text{Diagram 1: A 2-gon with a red side labeled } n \text{ and a blue side labeled } m. \text{ A dashed orange line labeled } F \text{ connects two vertices. A red line labeled } N \text{ and a blue line labeled } M \text{ are also shown.} \end{array} = \dim(\mathcal{M}) \quad \begin{array}{c} \text{Diagram 2: A simplified 2-gon with a red side labeled } n \text{ and a blue side labeled } m. \text{ A dashed orange line labeled } F \text{ connects two vertices. A red line labeled } N \text{ and a blue line labeled } M \text{ are also shown.} \end{array}$$

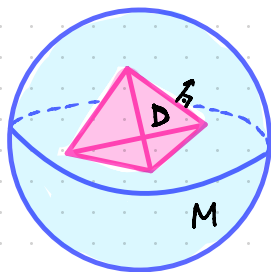
Back to state sums

EXAMPLE UNKNOTTED 3-BALL W/ DEFECT SPHERE

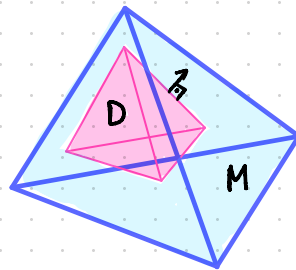


A solid ball w/ a hollow sphere
inside as defect.
No defect lines or vertices.

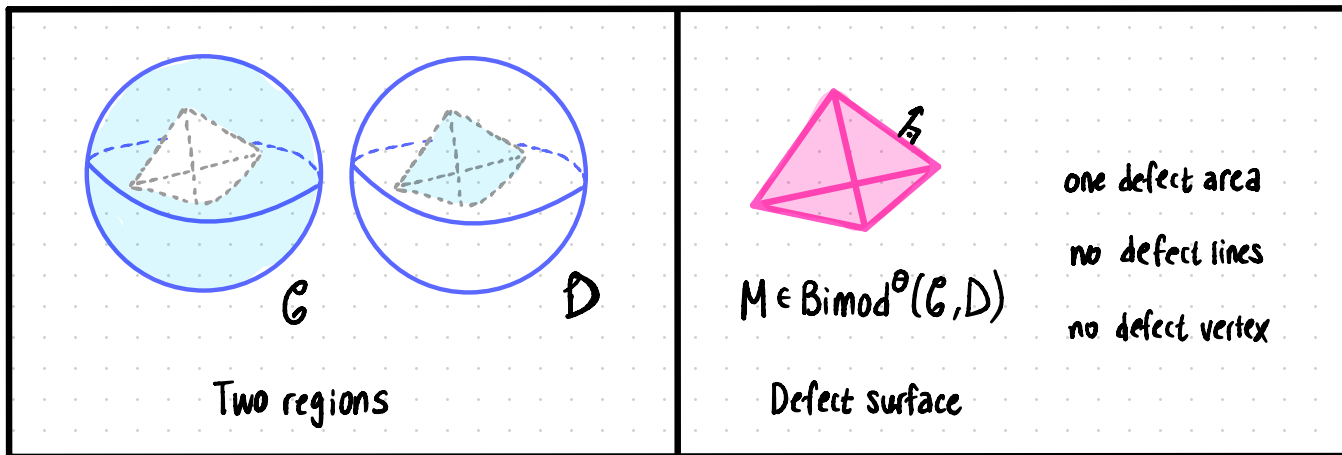
Assume the sphere is
unknotted, i.e. we can
PL Homeo to make the
sphere a tetrahedron.



or

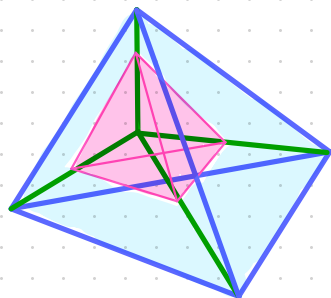


EXAMPLE UNKNOTTED 3-BALL W/ DEFECT SPHERE

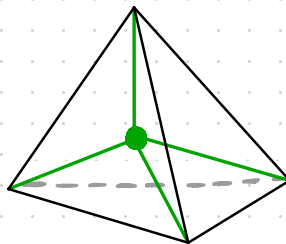


EXAMPLE UNKNOTTED 3-BALL W/ DEFECT SPHERE

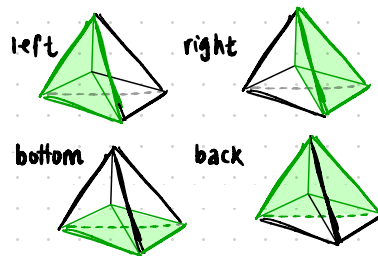
Pick
triangulation



$(M, \partial M, T, D)$

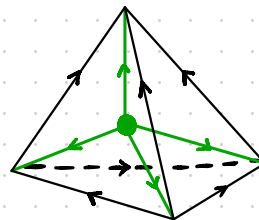
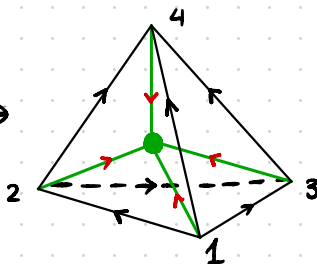
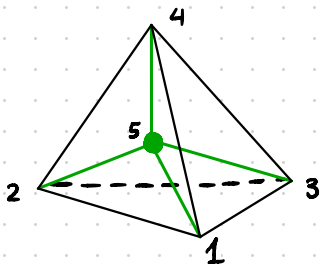


$(M, \partial M, T)$



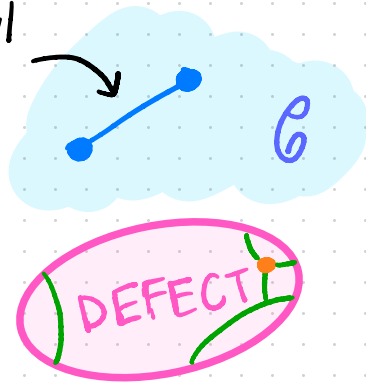
tetrahedra of triangulation

Orient

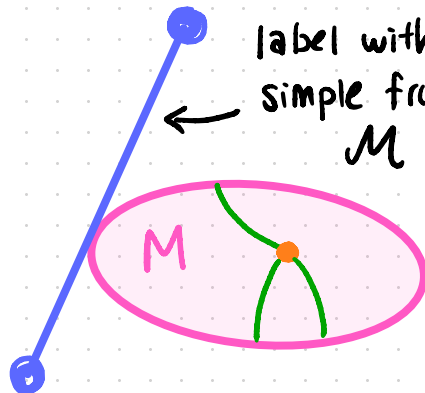


LABELING TRIANGULATIONS

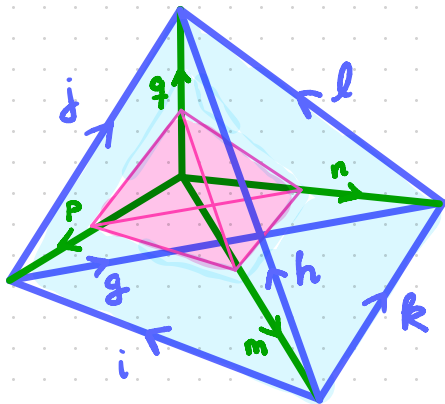
label w/
simple
from $\mathcal{G}$



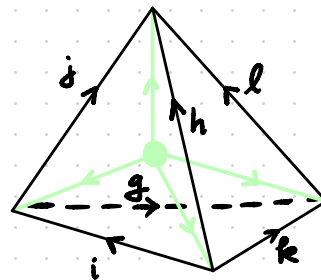
label with
simple from
 $\mathcal{M}$



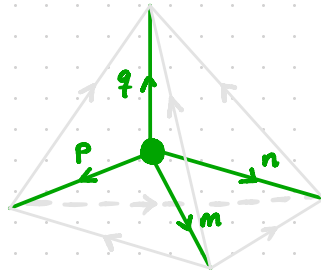
EXAMPLE UNKNOTTED 3-BALL W/ DEFECT SPHERE



from $\mathcal{G}$

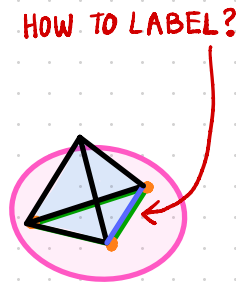
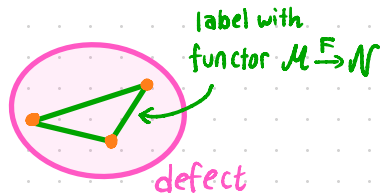
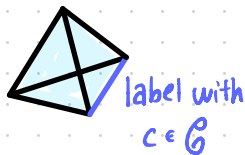


from $\mathcal{M}$



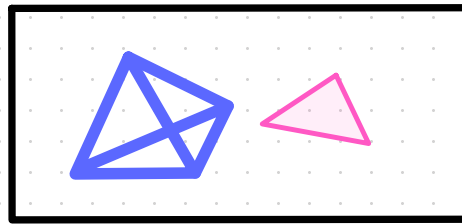
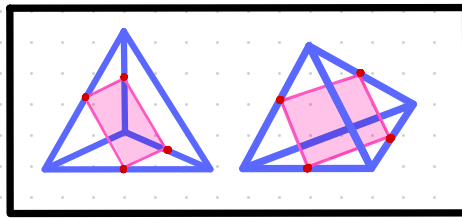
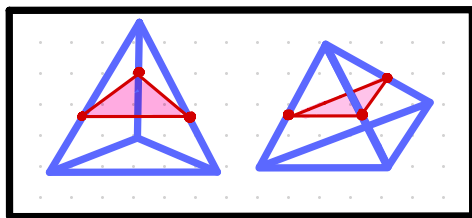
Big Idea: Adding the defect manifolds labeling shouldn't be too bad if the defect doesn't intersect the triangulation anywhere interesting.

e.g. don't want



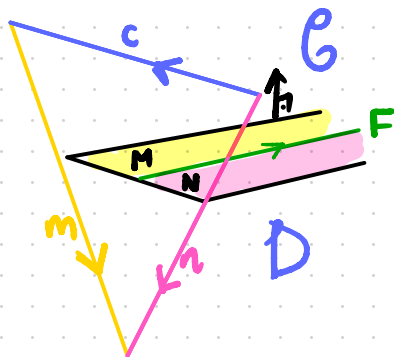
→ use generic transversal triangulations.

Proposition Any triangulation can be perturbed to a generic + transversal one.



ASSIGNING HOM-SPACES TO TRIANGLES

Modulo orientations, all triangles intersecting defect look like:



$$c, m, n, M \xrightarrow{F} N$$

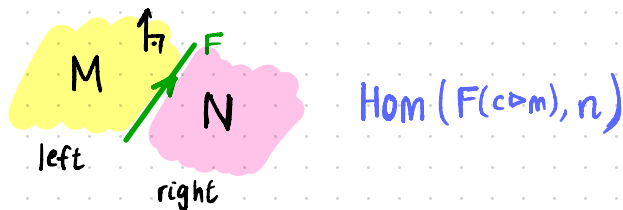
Combine the two edges w/ same direction

$$c \triangleright m, n, M \xrightarrow{F} N$$

Apply functor so objs are in same place

$$F(c \triangleright m), n$$

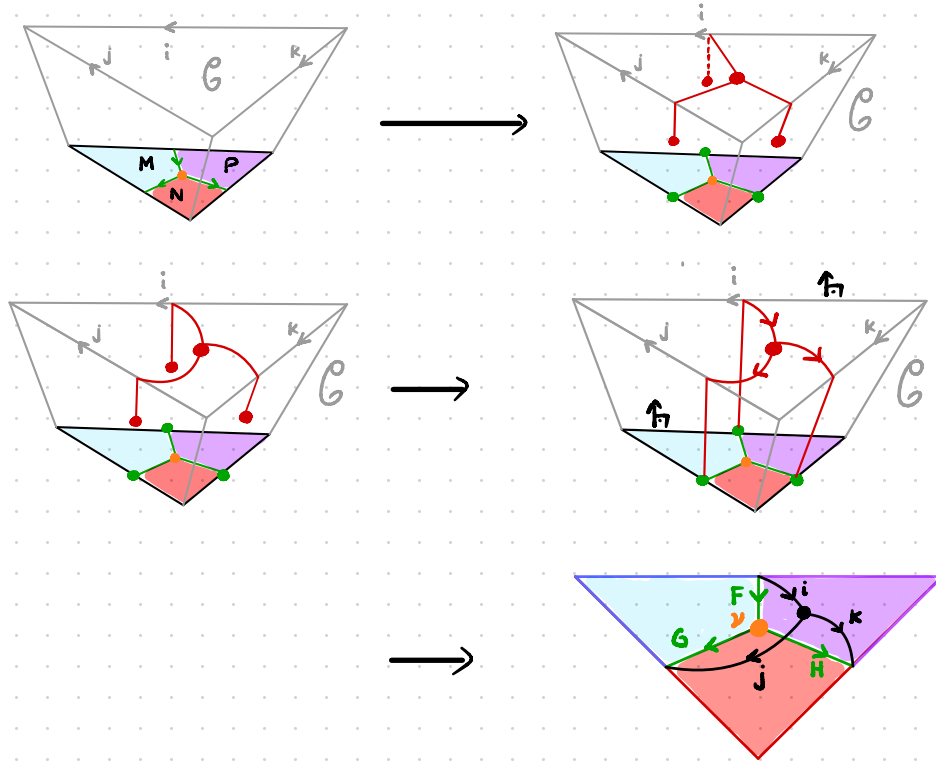
"The left one" wrt to defect normal is domain



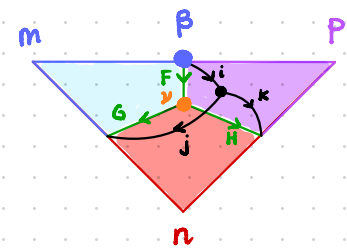
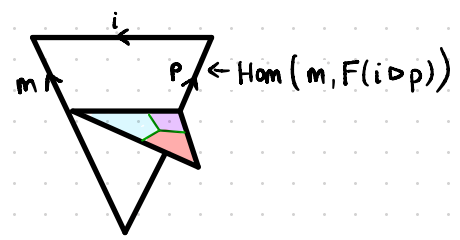
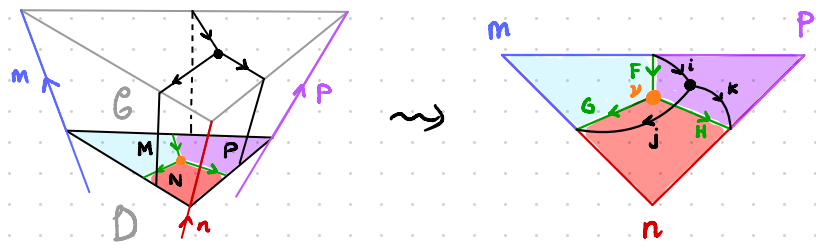
6j - Symbols

Draw dual graph
for edges that don't
touch defect

Curve, orient nicely
& project to defect

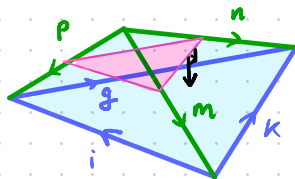
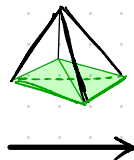
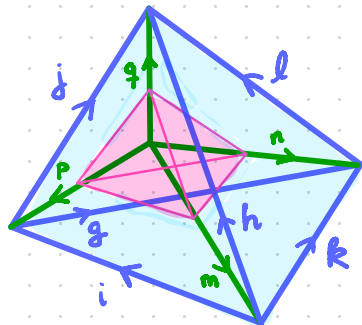


Label borders:

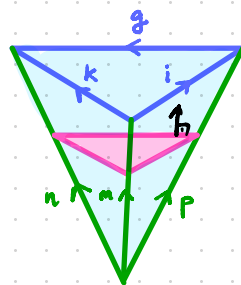


$$6j(\triangle) = ev(P)$$

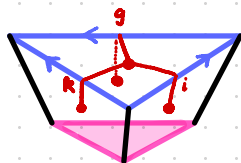
EXAMPLE UNKNOTTED 3-BALL W/ DEFECT SPHERE



rotate



draw dual edges
for edges not
touching defect

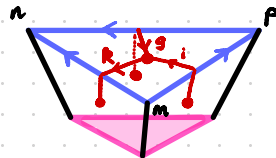


curve

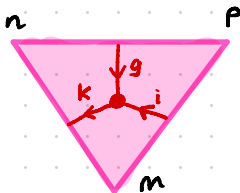
(not needed)

X

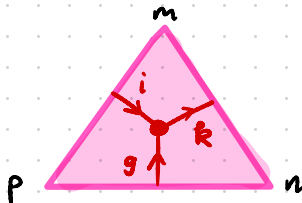
orient



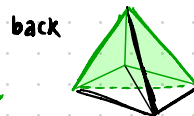
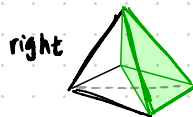
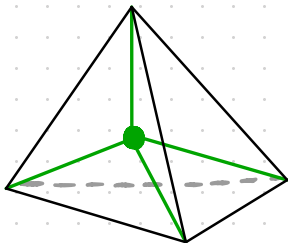
project to
defect



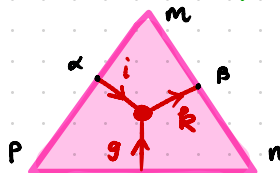
rotate



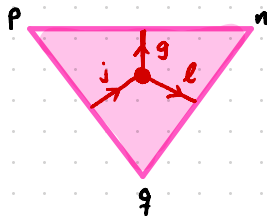
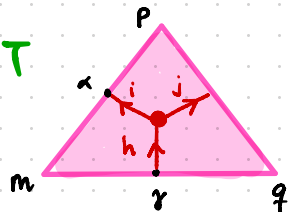
EXAMPLE UNKNOTTED 3-BALL W/ DEFECT SPHERE



BOTTOM

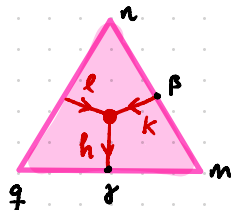


LEFT



BACK

RIGHT



STATE SUM W/ DEFECT

$$Z'(M, l_{\partial M}, b) := \frac{1}{\prod_{r \in R} \prod_{v \in V_r} \dim(G_r)} \sum_{\ell_i: \ell \in \mathcal{L}} \sum_{\text{Hom}(\Delta_i)} \prod_{t \in \text{Tet}} 6_j(t) \prod_{e \in E} \dim(\ell(e))$$

weight $\uparrow$ $\sum$ over all internal labelings $\uparrow$ 6_j $\uparrow$ weights

sum over all choices of projections + inclusions for internal triangles

To make it work with gluing, change $\dim(G_r), \dim(\ell(e)) \rightarrow \sqrt{\dim(G_r)}, \sqrt{\dim(\ell(e))}$ for boundary points + edges.

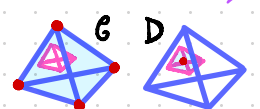
$$Z(M, l_{\partial M}, b) = \frac{1}{\prod_{r \in R} \prod_{v \in V_r} \dim(G_r)^{\epsilon(v)}} \sum_{\ell_i: \ell \in \mathcal{L}} \sum_{\text{Hom}(\Delta_i)} \prod_{t \in \text{Tet}} 6_j(t) \prod_{e \in E} \dim(\ell(e))^{\epsilon(e)}$$

$\epsilon(e) = \frac{1}{2}$ for ∂ edges

EXAMPLE UNKNOTTED 3-BALL W/ DEFECT SPHERE

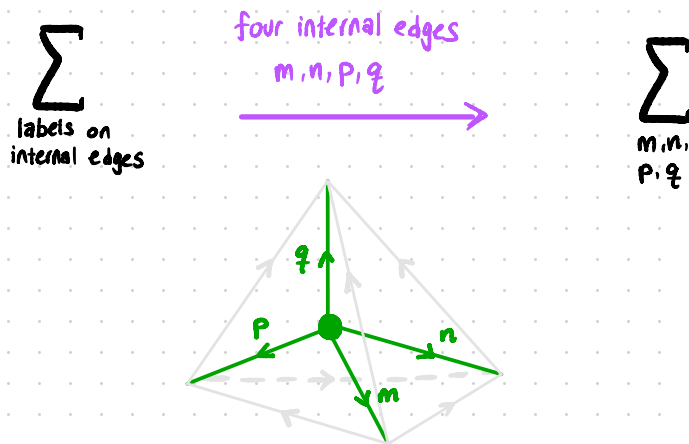
$$Z'(M, l_{\text{OM}}, b) := \frac{1}{\prod_{r \in R} \prod_{v \in V_r} \dim(G_r)} \sum_{\substack{\text{labels} \\ \text{on} \\ \text{internal} \\ \text{edges}}} \sum_{\substack{\text{assignments} \\ \text{of } p/j\text{'s} \\ \text{to internal} \\ \text{triangles}}} \prod_{\substack{\text{Tetrahedra} \\ t}} 6_j(t) \prod_{\substack{\text{edges} \\ e}} \dim(\ell(e))$$

Break it down:

$$\frac{1}{\prod_{r \in R} \prod_{v \in V_r} \dim(G_r)} \xrightarrow{\text{Two regions: } G, D \\ 4 \text{ vertices in } G \\ 1 \text{ vertex in } D} \frac{1}{\dim(G)^4 \dim(D)}$$


The diagram illustrates the decomposition of the product of dimensions of regions. It shows two tetrahedra, labeled G and D. Tetrahedron G is on the left and has 4 vertices. Tetrahedron D is on the right and has 1 vertex. A purple arrow points from the first fraction to the second, with the text 'Two regions: G, D' and '4 vertices in G, 1 vertex in D' above it.

$$Z'(M, l_{\text{om}}, b) := \frac{1}{\prod_{r \in R} \prod_{v \in V_r} \dim(G_r)} \sum_{\substack{\text{labels} \\ \text{on} \\ \text{internal} \\ \text{edges}}} \sum_{\substack{\text{assignments} \\ \text{of } p/q\text{'s} \\ \text{to internal} \\ \text{triangles}}} \prod_{\substack{\text{Tetrahedra} \\ t}} 6_j(t) \prod_{\substack{\text{edges} \\ e}} \dim(\ell(e))$$

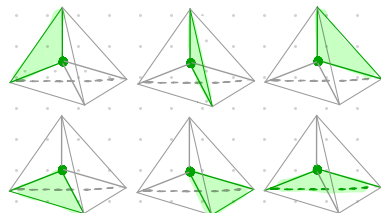


$$Z'(M, l_{\text{om}}, b) := \frac{1}{\prod_{r \in R} \prod_{v \in V_r} \dim(G_r)} \sum_{\substack{\text{labels} \\ \text{on} \\ \text{internal} \\ \text{edges}}} \sum_{\substack{\text{assignments} \\ \text{of } p/j\text{'s} \\ \text{to internal} \\ \text{triangles}}} \prod_t 6j(t) \prod_{\text{edges } e} \dim(l(e))$$

$$6j(t) \xrightarrow{\text{def}} = \text{ev}(P) =$$

$\uparrow$ tetrahedron


$$\sum_{\substack{\text{assignments} \\ \text{of } p/j\text{'s} \\ \text{to internal} \\ \text{triangles}}} \xrightarrow{\text{Six internal triangles}} \sum_{\alpha, \beta, \gamma, \delta, \sigma, \tau}$$



$$Z'(M, l_{\text{OM}}, b) := \frac{1}{\prod_{r \in R} \prod_{v \in V_r} \dim(\mathcal{C}_r)} \sum_{\substack{\text{labels} \\ \text{on} \\ \text{internal} \\ \text{edges}}} \sum_{\substack{\text{assignments} \\ \text{of } p/j\text{'s} \\ \text{to internal} \\ \text{triangles}}} \prod_{\substack{\text{Tetrahedra} \\ t}} 6_j(t) \prod_{\substack{\text{edges} \\ e}} \dim(\ell(e))$$

Recall "gluing sides" for polygon diagrams:

$$\sum_{\alpha} \text{[Diagram 1]} \cdot \text{[Diagram 2]} = \text{[Diagram 3]}$$

The diagram shows a gluing operation. On the left, a grey trapezoid with a blue top edge and red bottom edge is multiplied by a grey rectangle with a blue top edge and red bottom edge. The result is a single grey shape where the rectangle is attached to the right side of the trapezoid, sharing the red bottom edge.

$$\sum_{\substack{\text{assignments} \\ \text{of } p/j\text{'s} \\ \text{to internal} \\ \text{triangles}}} \prod_{\substack{\text{Tetrahedra} \\ t}} 6_j(t) = \sum_{\alpha, \beta, \gamma, \delta, \sigma, \tau} \text{[Diagram 1]} \stackrel{\text{glue}}{=} \sum_{\alpha, \beta, \gamma} \text{[Diagram 2]}$$

The diagram illustrates a gluing operation between two configurations of pink triangles. On the left, three triangles are arranged in a row, each with internal edges labeled with Greek letters (α, β, γ, δ, σ, τ) and red arrows. On the right, the result of gluing these triangles together is shown, where the internal edges are now part of a larger structure, and the sum is over the remaining external labels α, β, γ.

$$Z'(M, l_{OM}, b) := \frac{1}{\prod_{r \in R} \prod_{v \in V_r} \dim(G_r)} \sum_{\substack{\text{labels} \\ \text{on} \\ \text{internal} \\ \text{edges}}} \sum_{\substack{\text{assignments} \\ \text{of } p/j\text{'s} \\ \text{to internal} \\ \text{triangles}}} \prod_{\substack{\text{Tetrahedra} \\ t}} 6j(t) \prod_{\substack{\text{edges} \\ e}} \dim(l(e))$$

$$\prod_{\substack{\text{edges} \\ e}} \dim(l(e)) = \prod_{\substack{\text{edges} \\ \text{labelled w/} \\ G}} \dim(l(e)) \prod_{\substack{\text{edges} \\ \text{labelled w/} \\ \mathcal{M}}} \dim(l(e))$$

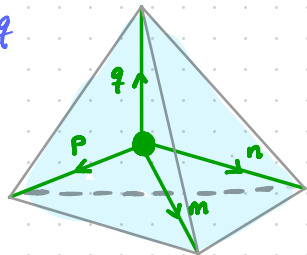
pull out and use

$$\dim(G) = \sum_{x \text{ in } G} \dim(x)^2$$

to cancel out the $\frac{1}{\dim(G)}$'s

inner edges m, n, p, q

$$= \dim(m) \dim(n) \dim(p) \dim(q)$$



$$Z' \left(\begin{array}{c} \text{Diagram of a tetrahedron with internal regions } D \text{ and } M \end{array} \right) = \frac{1}{\dim(D)} \sum_{\substack{m,n \\ p,q}} \sum_{\alpha,\beta,\gamma} \begin{array}{c} \text{Diagram of a triangle with internal structure} \end{array} \dim(m) \dim(n) \dim(p) \dim(q).$$

Recall "glue around a vertex" for polygon diagrams:

$$\sum_n \alpha \dim(n) \begin{array}{c} \text{Diagram of a vertex with four incident edges labeled } \alpha, \beta, \gamma, \delta \end{array} = \begin{array}{c} \text{Diagram of a vertex with two incident edges labeled } N, M \end{array}$$

$$= \sum_{\substack{m,n \\ p,q}} \sum_{\alpha,\beta,\gamma} \begin{array}{c} \text{Diagram of a triangle with internal structure and labels } \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \iota, \kappa, \lambda, \mu, \nu, \xi \end{array} \frac{\dim(m) \dim(n) \dim(p) \dim(q)}{\dim(D)}$$

$$= \sum_{m,n,q} \sum_{\beta,\gamma} \text{Diagram} \frac{\dim(m) \dim(n) \dim(q)}{\dim(D)}$$

The diagram is a pink diamond shape with vertices labeled β (top), γ (bottom), m (left), and n (right). Inside the diamond, there is a central red dot labeled μ . Four red arrows point from μ to the vertices: i to β , j to γ , g to m , and h to n . Additionally, there are red arrows forming a cycle: k from β to m , l from m to n , p from n to γ , and q from γ to β .

$$= \sum_{m,n,q} \sum_{\beta,\gamma} \text{Diagram} \frac{\dim(m) \dim(n) \dim(q)}{\dim(D)}$$

The diagram is identical to the one above, but with a yellow highlight on the top-right edge of the pink diamond, which connects the β and n vertices.

$$= \sum_{m, q, \gamma} \frac{\dim(m) \dim(q)}{\dim(D)} \cdot \text{Diagram}$$

Recall "gluing a 2-gon":

$$\sum_{m, n, \alpha} \dim(m) \dim(n) \cdot \text{Diagram} = \dim(\mathcal{M}) \cdot \text{Diagram}$$

$$= \frac{\dim(\mathcal{M})}{\dim(D)} \cdot \text{Diagram}$$

← Barret-Westbury
Gj for

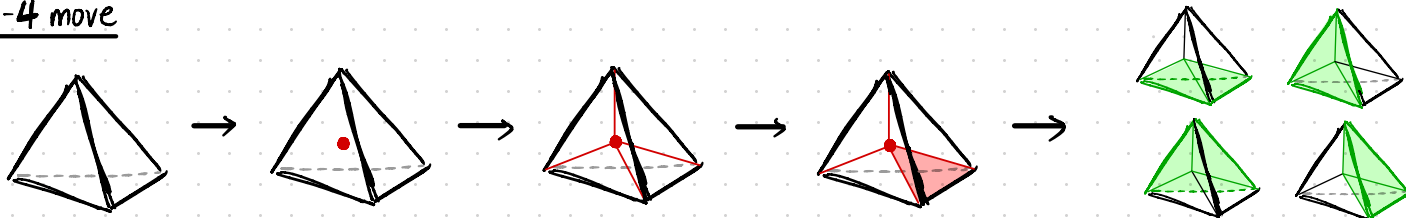
$$Z' \left(\text{Diagram 1} \right) = \frac{\dim(\mathcal{M})}{\dim(\mathcal{D})} Z' \left(\text{Diagram 2} \right)$$

The diagram on the left (Diagram 1) shows a light blue tetrahedron with a pink tetrahedron inscribed within it. The pink tetrahedron is smaller and is positioned such that its vertices are on the edges of the blue tetrahedron. The pink tetrahedron is labeled with a small 'z' and the blue tetrahedron is labeled with a small 'M'.

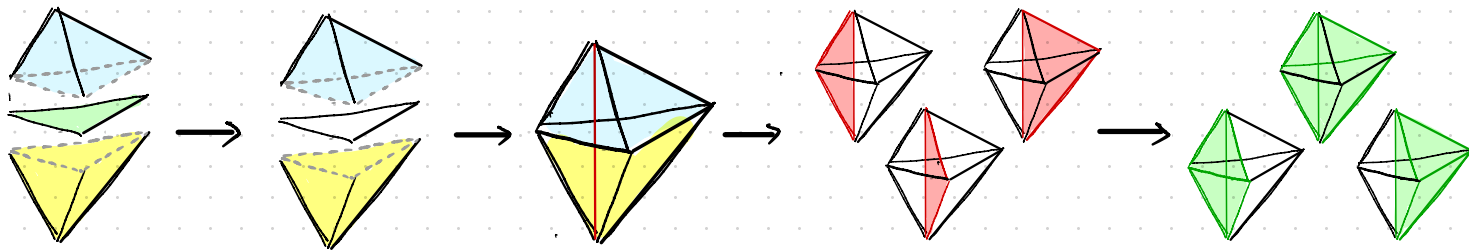
The diagram on the right (Diagram 2) shows a light blue tetrahedron with its diagonals drawn. The tetrahedron is labeled with a small 'M'.

TRIANGULATION INDEPENDENCE

1-4 move

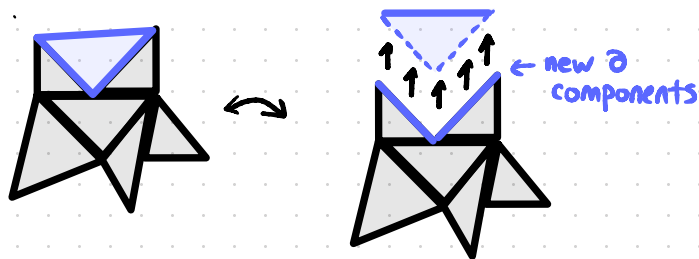


2-3 move

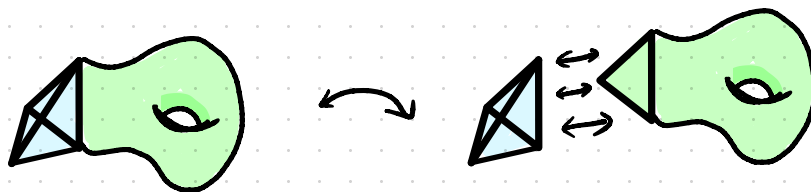


Elementary Shellings

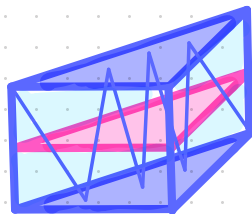
$n=2$



$n=3$



FINE NEIGHBORHOODS



generic
transversal
 $\cong [0,1] \times D$

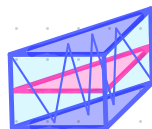
TRIANGULATION INDEPENDENCE

Idea: Use bistellar moves to refine triangulations & split into chunks that are all



no defect

or



fine ngbd w/
defect disk

Show the latter have triangulation + bistellar invariant state sums.

Glue everything back to diff. triangulations while checking state sum remains the same.